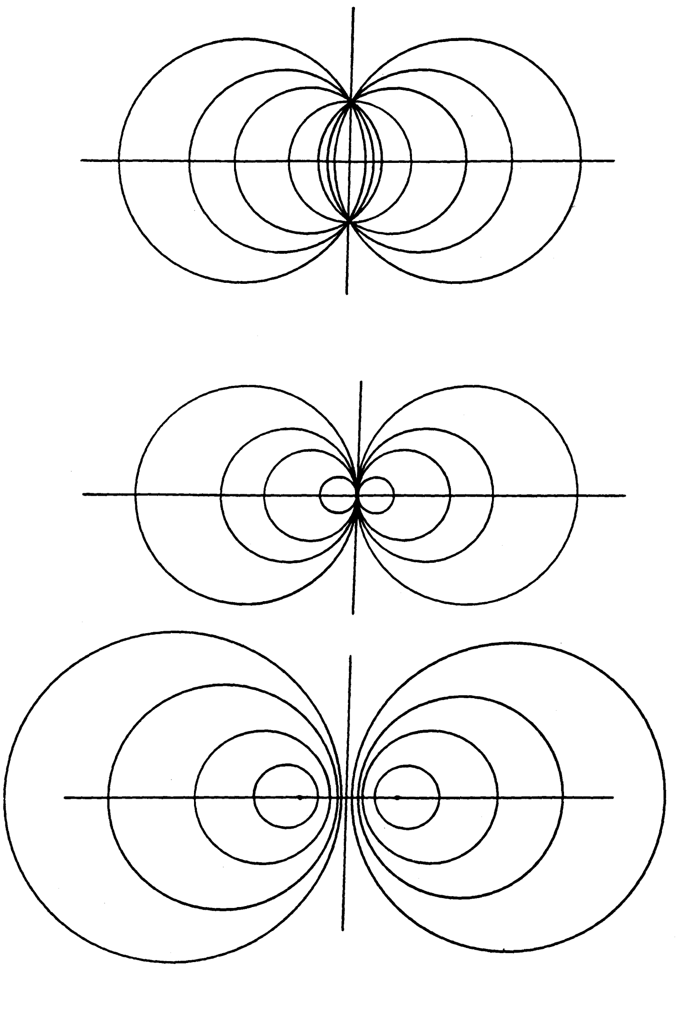
Vol. 66 No. 2 April 1993

MATHEMATICS MAGAZINE



- Circles, Vectors, and Linear Algebra
- Extended Pascal Triangles
- Graceful Configurations in the Plane

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Richard E. Pfiefer received his Ph.D. in 1982 from the University of California, Davis, with a thesis in geometry, written under the direction of G. Don Chakerian. He has been teaching at San Jose State University since 1983. He serves as a collaborating editor for the Problem Section of the *MONTHLY* and he recently completed service as an assistant editor for the *CMJ*. His research interests are in geometry, convexity, and the history of geometry. The present article evolved from the thesis written by his former master's student and now coauthor.

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Richard C. Bollinger has B.S., M.S., and Ph.D. degrees from the University of Pittsburgh. He began life as a classical analyst, but a variety of fortunate circumstances and collaborations led to his being able to also work in statistics, reliability theory, and, most recently, combinatorics. He did not invent the extended Pascal triangles, but he did discover them independently while working on the combinatorics of a problem of reliability theory, and has been interested ever since in the development of their properties and applications. He is Professor of Mathematics at Penn State at Erie, The Behrend College.

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The *MATHEMATICS MAGAZINE* (ISSN 0025–570X) is published by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, D.C. 20036 and Montpelier, VT, bimonthly except July/August.
The annual subscription price for the MATHEMATICS MAGAZINE to an individual member of the Association is $16 included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $64. Student and unemployed members receive a 66% dues discount; emeritus members receive a 50% discount; and new members receive a 40% dues discount for the first two years of membership.) The nonmember/library subscription price is $68 per year.
Subscription correspondence and notice of change of address should be sent to the Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036. Microfilmed issues may be obtained from University Microfilms International, Serials Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

Advertising correspondence should be addressed to Ms. Elaine Pedreira, Advertising Manager, The Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036.

Second class postage paid at Washington, D.C. and additional mailing offices.

Postmaster: Send address changes to Mathematics Magazine Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036-1385.

PRINTED IN THE UNITED STATES OF AMERICA

ARTICLES

Circles, Vectors, and Linear Algebra

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Some of the named theorems in the elementary geometry of circles can be obtained by treating circles as vectors and using some basic ideas from linear algebra. The methods we will describe were independently developed in the 1860s and 1870s by Clifford [**3**, **4**], Frobenius [**9**], and Darboux [**7**]. These ideas were expanded upon by Cox [**6**] in 1883 and Lachlan [**11**] in 1886 and many of our theorems are adapted from their articles. Other sources include the books by Coolidge [**5**], Forder [**8**], and Pedoe [**16**, **17**]. Pedoe's books are highly recommended for their introduction to the subject.

1. Circles as Vectors

We assume the reader is familiar with the concepts of vectors in $\mathbb{R}^n$ and their inner products, systems of linear equations and their determinants, and plane analytic geometry. To save space we will leave the algebra verifications and simplifications to the reader whenever possible. This section begins with a summary of Pedoe's [**16**] introduction to the subject.

The set of points satisfying the equation $C(x, y) = \alpha(x^2 + y^2) - 2\beta x - 2\gamma y + \delta = 0$, with $\alpha \neq 0$, is called a *proper circle* and is denoted by C. We will assign it the coordinate vector $C(\alpha, \beta, \gamma, \delta)$ in $\mathbb{R}^4$. Since any nonzero multiple of the equation represents the same circle, we usually restrict ourselves to the normalized equation of C: $x^2 + y^2 - 2fx - 2gy + k = 0$ and its associated coordinate vector $C(1, f, g, k)$. Note that all non-zero scalar multiples of this coordinate represent the same circle, so each proper circle corresponds to a line through the origin in $\mathbb{R}^4$. Such a line is a "point" of real projective 3-space, $\mathbb{R}P^3$, so we have a one-to-one correspondence between the circles in the plane and a subset of "points" in $\mathbb{R}P^3$. Each "point" of $\mathbb{R}P^3$ has a homogeneous coordinate consisting of all nonzero scalar multiples of some fixed 4-tuple (a, b, c, d) in $\mathbb{R}^4$.

Using the normalized equation of C, the radius r is defined by $r^2 = f^2 + g^2 - k$. If $r^2 > 0$, we say C is a *real circle*. If $r^2 = 0$, the locus of points satisfying the equation is a single point and C is called a *point circle*. If $r^2 < 0$, then there are no real points (x, y) satisfying the equation and we say C is a *virtual circle*.

When $\alpha = 0$ and $\beta^2 + \gamma^2 > 0$, the equation $L(x, y) = \alpha(x^2 + y^2) - 2\beta x - 2\gamma y + \delta = 0$ represents an ordinary line L in the plane and has associated $\mathbb{R}^4$ vector (or homogeneous $\mathbb{R}P^3$ coordinate) $L(0, \beta, \gamma, \delta)$. In order to use some convenient formulas from analytic geometry, we restrict ourselves to the normalized equation of L: $-2fx - 2gy + k = 0$, where $f^2 + g^2 = 1$, and its coordinate $L(0, f, g, k)$.

If $\alpha=\beta=\gamma=0$ and $\delta\neq 0$ our equation becomes $L(x,y)=\delta=0$. No real points (x,y) satisfy this equation and it is convenient to associate this equation and its coordinate $(0,0,0,\delta)$ with the *line at infinity*. This equation and coordinate are normalized by taking $\delta=1$.

Note that the set of circles and lines in the plane (including virtual circles and the line at infinity) is in one-to-one correspondence with the "points" of $\mathbb{R}P^3$.

2. The Inner Product of Two Circles

For two vectors $V_i(\alpha_i,\beta_i,\gamma_i,\delta_i)$, $i=1,2$, in $\mathbb{R}^4$ we define the *inner product*

$$V_1\circ V_2=\beta_1\beta_2+\gamma_1\gamma_2-(\delta_1\alpha_2+\delta_2\alpha_1)/2.$$

This induces an inner product on the set of all circles and lines in the plane by its value on the normalized coordinate representatives of the circles and lines.

Let $C_i(1,f_i,g_i,k_i)$, $L_i(0,f_i,g_i,k_i)$, $E(0,0,0,1)$ be the normalized coordinate representatives of circles, lines and the line at infinity. The following facts are easily verified.

i. $C_1\circ C_1=r_1^2$.
ii. $C_1\circ C_2=f_1f_2+g_1g_2-(k_1+k_2)/2=(r_1^2+r_2^2-d^2)/2$, where d is the distance between the centers of C_1 and C_2. In particular, if C_1 intersects C_2, then $C_1\circ C_2=r_1r_2\cos\theta$, where θ, the angle between C_1 and C_2, is the (smaller) angle between the two radius vectors at the point of intersection. See FIGURE 1.

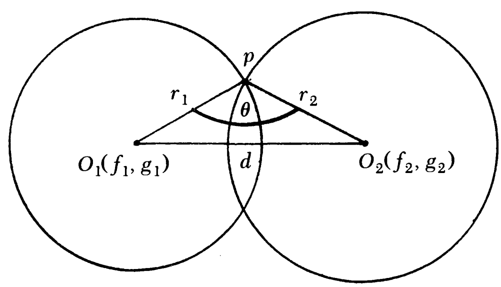

FIGURE 1

iii. Circle C_1 is orthogonal to C_2 if and only if the angle between C_1 and C_2 is a right angle if and only if $C_1\circ C_2=0$ (FIGURE 2).
iv. $C_1\circ C_2=\pm r_1r_2$ if and only if C_1 and C_2 are tangent.
v. If C_1 is a point circle (i.e. $r_1^2=f_1^2+g_1^2-k_1=0$), then $C_1\circ C_2=-(d^2-r_2^2)/2$; $d^2-r_2^2$ is known as the *power* of point C_1 with respect to circle C_2.
vi. A point circle C_1 is on circle C_2 if and only if $C_1\circ C_2=0$.
vii. Two point circles C_1 and C_2 satisfy $C_1\circ C_2=-d^2/2$, where d is the distance between the points.

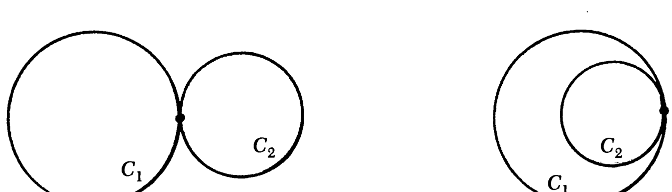

FIGURE 2

viii. $C_1 \circ L_2 = f_1 f_2 + g_1 g_2 - k_2/2 = -D$, so that D is the signed (or directed) distance from the center of C_1 to the line L_2: $-2f_2 x - 2g_2 y + k_2 = 0$. See FIGURE 3, where the normal vector to L_2 has components $(-2f_2, -2g_2)$, so that D is negative in this figure.

ix. Line L_2 is a diameter of C_1 if and only if $L_2 \circ C_1 = 0$. A point circle C_1 is on line L_2 if and only if $C_1 \circ L_2 = 0$.

x. $C_1 \circ E = -1/2$.

xi. $L_1 \circ E = 0$, i.e. all lines are orthogonal to the line at infinity.

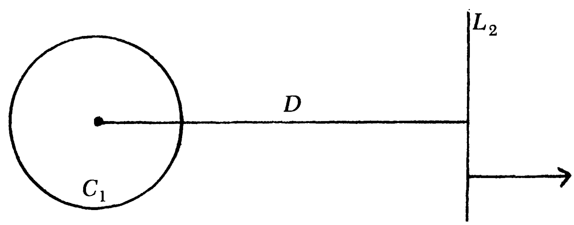

FIGURE 3

3. Linear Combinations of Circles

Let $C_i(1, f_i, g_i, k_i)$ $i = 1, 2$ be two circles. Then, for real numbers a_1, a_2 with $a_1 + a_2 \neq 0$, $C = (a_1 C_1 + a_2 C_2)/(a_1 + a_2)$ is a (normalized) circle with center

$$\left(\frac{a_1 f_1 + a_2 f_2}{a_1 + a_2}, \frac{a_1 g_1 + a_2 g_2}{a_1 + a_2} \right).$$

If $a_1 + a_2 = 0$, $C = a_1 C_1 + a_2 C_2$ is a line, called the *radical axis* of C_1 and C_2. The set of circles $C = a_1 C_1 + a_2 C_2$ (a_1, a_2 not both 0) is called a *coaxal system of circles*. See FIGURE 4. As a set of vectors in $\mathbb{R}^4$, the coaxal system is the set of all vectors that are dependent on C_1 and C_2, i.e., it is the span of C_1 and C_2.

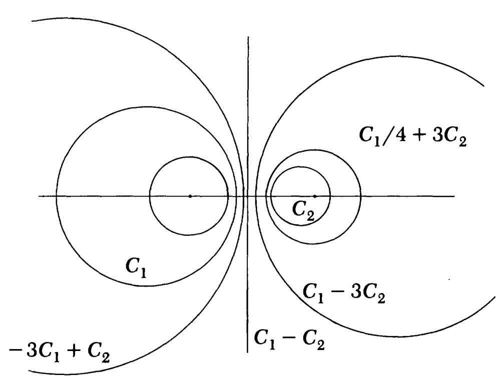

FIGURE 4

A point circle P is on the radical axis $C_1 - C_2$ if $P \circ (C_1 - C_2) = 0$, or $P \circ C_1 = P \circ C_2$. That is, the radical axis is the locus of all points P having the same power with respect to C_1 and C_2. It is easy to show that P has the same power with respect to every circle in the system. If a circle C_3 is orthogonal to both C_1 and C_2, then $C_3 \circ (C_1 - C_2) = C_3 \circ C_1 - C_3 \circ C_2 = 0 - 0 = 0$, so the radical axis is a diameter of every such circle C_3. A point P is on both C_1 and C_2 if and only if P is on every circle of the system. In FIGURE 5 we illustrate intersecting, tangent, and nonintersecting coaxal systems (or linear combinations) of circles.

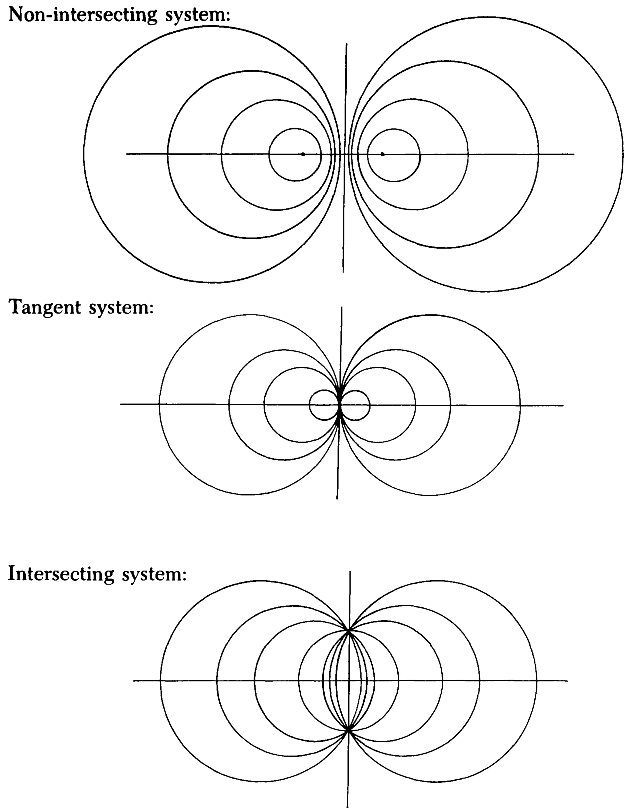

FIGURE 5

Our first theorem, whose proof illustrates one application of these vector methods, is described by Johnson [10] as "an open sesame to a number of theorems and developments" in the geometry of circles.

CASEY'S POWER THEOREM. *For any point P and circles C_1 and C_2 let D be the signed distance from P to the radical axis, $C_1 - C_2$, and let d be the distance between the centers of C_1 and C_2 as shown in* FIGURE 6. *Then*

$$2Dd = \left(d_1^2 - r_1^2\right) - \left(d_2^2 - r_2^2\right).$$

Proof. $-D = P \circ \dfrac{C_1 - C_2}{d} = (P \circ C_1 - P \circ C_2)/d = (-(d_1^2 - r_1^2) + (d_2^2 - r_2^2))/2d.$

Since the inner product of a circle with itself is the square of its radius, corresponding to the circles C_1 and C_2 we have the "unit vectors" C_1/r_1 and C_2/r_2. In analogy with ordinary vectors, the "internal and external angle bisectors" are defined as $I = C_1/r_1 + C_2/r_2$ and $E = C_1/r_1 - C_2/r_2$. I and E are called the *internal* and *external circles of antisimilitude* of the circles C_1 and C_2. It follows from viii and ix that the centers of E and I lie at the points where the internal tangents and external tangents of C_1 and C_2 meet. See FIGURE 7, where only the external tangent lines are drawn.

As an aside on the inversion transformation of circles and lines, we mention that the reflection of vector U in V (or parallel to V) is given by

$$U' = \pm 2V(U \circ V)/(V \circ V) \mp U,$$

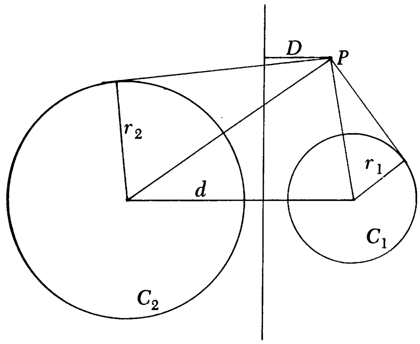

FIGURE 6

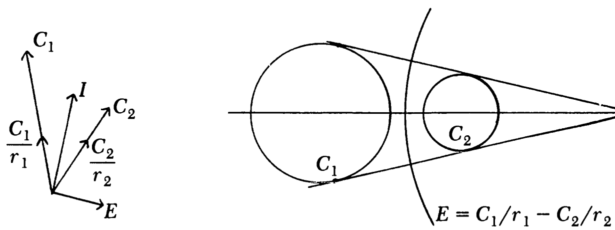

FIGURE 7

and this is the formula for the inversion of circle U in circle V, using the inner product for circles [1]. As an example, the reflection of vector C_1 in vectors I, E gives vectors C_2, $-C_2$ (up to a scalar multiple), so the inversion of circle C_1 in circles I, E gives the circle C_2.

Three circles, C_i, are linearly independent if $\Sigma x_i C_i = 0$ has only the $x_i = 0$ solution. That is, three circles are independent if no one is in the coaxal system of the other two.

MENELAUS' THEOREM (Vector Form). *Given three linearly independent circles* C_i *and letting* $D_1 = a_1C_2 + b_1C_3$, $D_2 = a_2C_3 + b_2C_1$, $D_3 = a_3C_1 + b_3C_2$, *then* D_1, D_2, D_3 *are dependent (i.e. in a coaxal system) if and only if* $a_1a_2a_3 = -b_1b_2b_3$.

Proof. There are nonzero x_i satisfying $\Sigma x_i D_i = 0$ if and only if the homogeneous system

$$\begin{bmatrix} 0 & b_2 & a_3 \\ a_1 & 0 & b_3 \\ b_1 & a_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a nonzero solution x_i, or equivalently, the determinant $a_1a_2a_3 + b_1b_2b_3$, is zero.

COROLLARY. *Let* C_i *be three linearly independent circles and let* I_{ij} *and* E_{ij} *be the internal and external circles of antisimilitude of* C_i *and* C_j. *Then each of the four sets of circles* $\{E_{12}, E_{23}, E_{31}\}, \{E_{12}, I_{23}, I_{31}\}, \{I_{12}, E_{23}, I_{31}\}, \{I_{12}, I_{23}, E_{31}\}$ *is dependent, i.e. coaxal.*

Proof. $E_{12} = C_1/r_1 - C_2/r_2$, $E_{23} = C_2/r_2 - C_3/r_3$, $E_{31} = C_3/r_3 - C_1/r_1$ and

$$\frac{1}{r_1r_2r_3} = -\frac{1}{(-r_1)(-r_2)(-r_3)}.$$

The other cases are similar.

As a consequence of this, we have the

MONGE THREE CIRCLE THEOREM. *The centers of the circles of the four sets above are collinear.*

This theorem is illustrated in FIGURE 8 for the first set. Note that the collinear points are determined by the internal and external common tangents of the circles C_i.

The following observation is needed in the next theorem and the subsequent discussion. Let C_1 be orthogonal to C_2, with centers O_1 and O_2, and let the *average circle* $A=(C_1+C_2)/2$. Then segment O_1O_2 is a diameter of A, because $O_i \circ C_i = r_i^2/2$ and $O_i \circ C_j = -r_i^2/2$, so $O_i \circ A = 0$ (FIGURE 9).

THEOREM. *Let C_1, C_2, C_3 be three mutually orthogonal circles; suppose that the circles D_1, D_2, D_3 (normalized) defined as in Menelaus' Theorem are dependent (i.e. coaxal), and let $A_i=(C_i+D_i)/2$, $i=1,2,3$ be the average circles. Then the circles A_i are dependent.*

The proof is similar to that of Menelaus' Theorem.

If the centers of the circles C_i in the theorem above are the three noncollinear points O_i, then the centers, P_i, of the circles D_i are collinear and lie on the sides of the triangle $\Delta O_1O_2O_3$ as shown in FIGURE 10a. Such a configuration of four lines and their six points of intersection is called a *complete quadrilateral*. The line segments O_1P_1, O_2P_2, O_3P_3 are called the *diagonals* of the complete quadrilateral. Notice that C_i (with center O_i) is orthogonal to D_i (with center P_i) so the average circle A_i has diameter O_iP_i. This is illustrated in FIGURE 10b.

Thus the theorem above yields the

GAUSS-BODENMILLER THEOREM. *The circles A_1, A_2, A_3 (described above) with diameters O_1P_1, O_2P_2, O_3P_3, the diagonals of a complete quadrilateral, are coaxal.*

The centers of these coaxal circles lie on a line, so we have another consequence known as

NEWTON'S THEOREM. *The midpoints of the diagonals of a complete quadrilateral are collinear.*

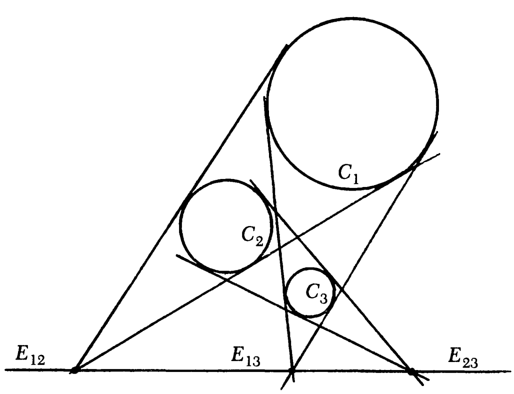

FIGURE 8

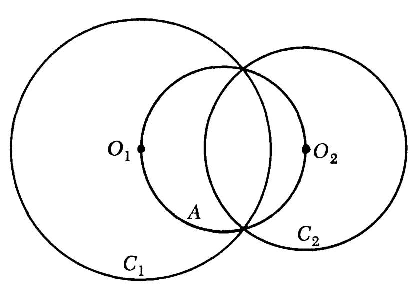

FIGURE 9

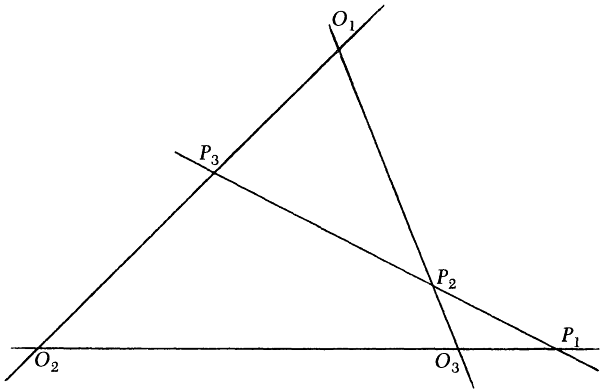

FIGURE 10a

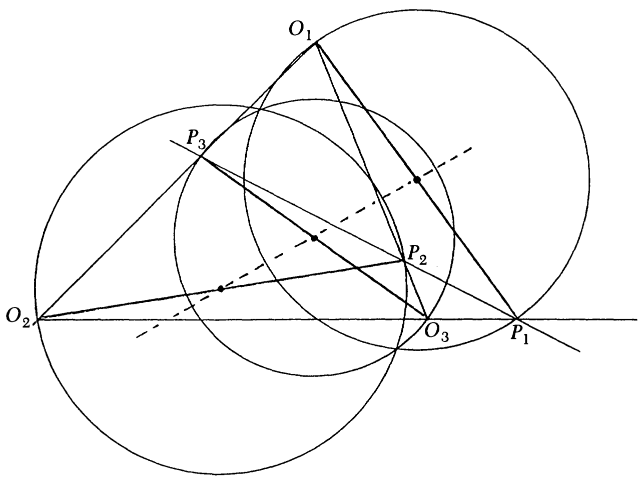

FIGURE 10b

4. The Gram Determinant

Given a dependent set of vectors $\{V_1, \ldots, V_n\}$ there is a nonzero solution to $\Sigma x_i V_i = 0$. Taking the inner product successively with vectors $W_1, \ldots, W_n$ gives a system of homogeneous linear equations with nonzero solution x_i:

$$\begin{bmatrix} W_1 \circ V_1 & \cdots & W_1 \circ V_1 \\ \vdots & & \vdots \\ W_n \circ V_n & \cdots & W_n \circ V_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The determinant of the matrix, denoted $G(V_i; W_i)$, must be zero. If $W_i = V_i$, we denote the determinant as $G(V_i)$. This is called the *Gram determinant* of the vectors V_i.

Given four mutually orthogonal vectors C_i with radius r_i, $i = 1, 2, 3, 4$, and letting $C_5 = E(0, 0, 0, 1)$, the five vectors C_i must be dependent, and $G(C_i) = 0$ (called the Frobenius Identity by Coolidge [5]) yields

$$1/r_1^2 + 1/r_2^2 + 1/r_3^2 + 1/r_4^2 = 0.$$

This formula was given by Clifford [9] in 1865. Note that at least one of the circles must be virtual ($r_i^2 < 0$) and at least one must be real ($r_i^2 > 0$). In fact, it can be shown that given four mutually orthogonal circles, exactly three must be real and one must be virtual. Furthermore, their centers form an *orthocentric system*. That is, each center point is the orthocenter (point of concurrence of the altitudes) of the triangle formed by the other three points.

Taking the same set of circles C_i (with centers P_i) and letting $D_i = C_i$ for $1 \le i \le 4$ and D_5 be any line, then $G(C_i; D_i) = 0$ yields

$$\frac{P_1D_5}{r_1^2} + \frac{P_2D_5}{r_2^2} + \frac{P_3D_5}{r_3^2} + \frac{P_4D_5}{r_4^2} = 0,$$

where P_iD_5 is the signed distance from point P_i to line D_5. If $D_5 = P$ is a point, then $G(C_i; D_i) = 0$ yields

$$\frac{P_1 \circ C_5}{r_1^2} + \frac{P_2 \circ C_5}{r_2^2} + \frac{P_3 \circ C_5}{r_3^2} + \frac{P_4 \circ C_5}{r_4^2} = 1.$$

Recall that $P \circ C_i = -(d_i^2 - r_i^2)/2$, where d_i is the distance from P to P_i. Lachlan [11] obtained the above two results. We suggest the reader sketch the circles C_i and select various points and lines for D_5 to obtain some special cases of these formulas.

Suppose we have four circles C_i (with centers O_i) orthogonal to a fixed circle C. Then the C_i, as vectors, are dependent and

$$0 = G(C_i) = \frac{1}{4}\begin{vmatrix} 1 & f_1 & g_1 & c_1 \\ 1 & f_2 & g_2 & c_2 \\ 1 & f_3 & g_3 & c_3 \\ 1 & f_4 & g_4 & c_4 \end{vmatrix}^2.$$

Setting the determinant on the right side equal to 0 and expanding along the last column we obtain

$$-c_1\Delta O_2O_3O_4 + c_2\Delta O_1O_3O_4 - c_3\Delta O_1O_2O_4 + c_4\Delta O_1O_2O_3 = 0.$$

This is known as *Harvey's Formula* [18] and we can visualize the product in the first term as shown in FIGURE 11, where $c_1 = t_1^2$ and t_1 is the tangent distance from the origin to circle C_1.

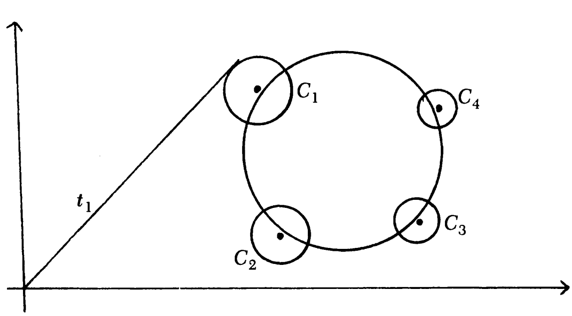

FIGURE 11

If the circles C_i shrink to point circles on C, then Harvey's Formula becomes *Feuerbach's Relation*:

$$d_1^2\Delta C_2C_3C_4 + d_3^2\Delta C_1C_2C_4 = d_2^2\Delta C_1C_3C_4 + d_4^2\Delta C_1C_2C_3,$$

where d_i is the distance from the origin to point C_i. See FIGURE 12. In the last two formulas the origin can be replaced by any point P in the plane and c_i and d_i will represent the power of P with respect to the circles C_i.

Let a set of point circles C_i that are concyclic (lie on the same circle) form a convex quadrilateral. Then $G(C_i)=0$, after simplification, yields $d_{13}d_{24}=d_{12}d_{34}+d_{23}d_{14}$. This, of course, is *Ptolemy's Theorem*. See FIGURE 13.

Let C_i, $i=1,2,3,4$, be any four points in the plane, $C_5=E(0,0,0,1)$, and d_{ij} be the distance between C_i and C_j $(i,j\neq 5)$. Then $G(C_i)=0$ gives

$$\begin{vmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 & 1 \\ d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 & 1 \\ d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 & 1 \\ d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

This is known as *Euler's Relation* and the derivation given above is essentially Cayley's [**2**]. This has an interesting corollary. Let D_1 be the circumcircle of the point circles C_2, C_3, C_4, D_2 that of C_1, C_3, C_4, and similarly for D_3 and D_4, and let $D_5=E(0,0,0,1)$. Then $G(C_i; D_i)=0$ gives

$$\frac{1}{C_1\circ D_1}+\frac{1}{C_2\circ D_2}+\frac{1}{C_3\circ D_3}+\frac{1}{C_4\circ D_4}=0.$$

Recall that $C_i\circ D_i>0$ if and only if the point C_i is inside the circumcircle D_i of the other three points. From the relation above, it follows that given four points in the plane, at least one point must be inside the circumcircle of the other three and at least one point must be outside the circumcircle of the other three.

For the next theorem we recall that a triangle has four circles touching all three of its side lines: the inscribed circle, or incircle, and the three escribed circles, or excircles, each of which is tangent to one side and the other two sides produced. The *nine-point circle* of a triangle passes through the feet of the altitudes, the midpoints of the sides, and the midpoints of the segments from the orthocenter to the vertices.

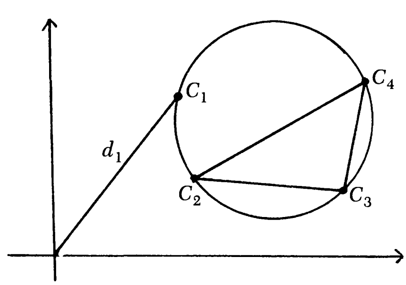

FIGURE 12

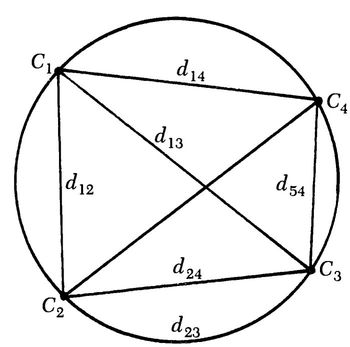

FIGURE 13

One of the celebrated theorems of the 19th century is

FEUERBACH'S THEOREM. *The incircle and the excircles of a triangle are tangent to the nine-point circle.*

To prove this, let C_0 be the incircle of $\Delta A_1A_2A_3$, let C_1, C_2, C_3 be the midpoints of sides a_1, a_2, a_3, and let C_4 be the nine-point circle of $\Delta A_1A_2A_3$. (See FIGURE 14.)

Then $G(C_i) = 0$ gives $(C_0 \circ C_4)^2 = (r_0 r_4)^2$, i.e. C_0 is tangent to C_4. Similarly, we can show that the nine-point circle, C_4, is tangent to each of the excircles.

Referring to FIGURE 14, let C_0 be the incircle of triangle $\Delta A_1 A_2 A_3$, C_i the excircles, D_i the side opposite vertex A_i, and let $D_4 = E(0,0,0,1)$. Then $G(C_i; D_i) = 0$ gives the relationship between the reciprocals of the inradius and exradii:

$$\frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

Let C_i, $i = 1,2,3,4$, be four mutually externally tangent circles and $C_5 = E$, then $G(C_i) = 0$ yields $\Sigma 1/r_i^2 = \Sigma_{i \neq j} 1/r_i r_j$, a formula obtained by Steiner in 1826 [5]. Pedoe [15] reports that this formula was rediscovered by Beecroft in 1842, then rediscovered again in the form $2\Sigma 1/r_i^2 = (\Sigma 1/r_i)^2$ by Soddy in 1936, and for some time was known as Soddy's Theorem. The theorem was recently traced back to its appearance in a 1643 letter of Descartes to the Queen of Bohemia. So the theorem is now known as *Descartes' Circle Theorem* (FIGURE 15).

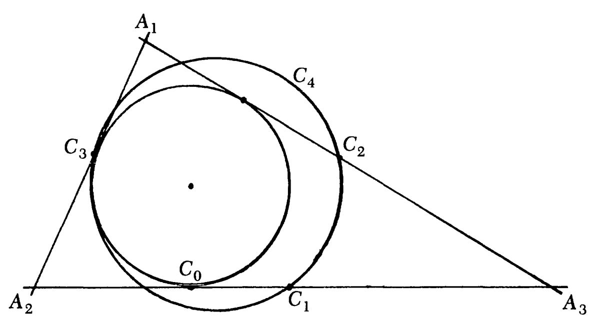

FIGURE 14

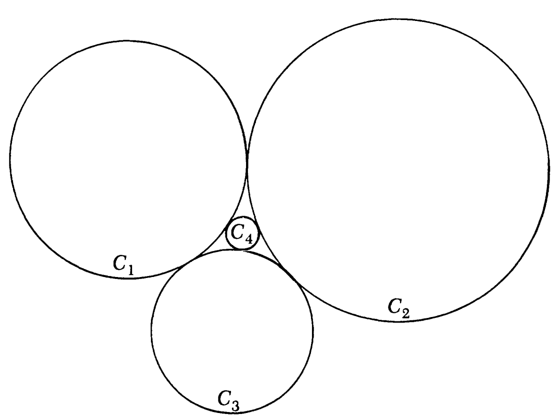

FIGURE 15

5. Change of Basis; Stereographic Projection

We now give a visual interpretation of the homogeneous coordinate representation of a planar circle using stereographic projection after an appropriate change of basis. (Pedoe [16, 17] provides another visualization using a paraboloid of revolution.) Our current, nonorthogonal basis (with corresponding equations) is:

$(1,0,0,0)$	$(0,1,0,0)$	$(0,0,1,0)$	$(0,0,0,1)$
$x^2+y^2=0$	$-2x=0$	$-2y=0$	
origin	y-axis	x-axis	line at infinity

We will choose the new orthogonal basis to be:

y-axis	x-axis	unit circle	virtual unit circle
$2x=0$	$2y=0$	$x^2+y^2-1=0$	$x^2+y^2+1=0.$

Given $C_i(a_i, b_i, c_i, d_i)$ with respect to the new basis, then $C_1 \circ C_2 = a_1a_2 + b_1b_2 + c_1c_2 - d_1d_2$, the four-dimensional Minkowski space-time inner product. If a point $P(x,y)$ has coordinates $P(a,b,c,d)$ with respect to the new basis, then these are called the *tetracyclic coordinates* of the point P. A point $P(x,y)$ is on circle $C(a,b,c,d)$ (new basis) if and only if

$$a(2x)+b(2y)+c(x^2+y^2-1)+d(x^2+y^2+1)=0,$$

or

$$a\left(\frac{2x}{x^2+y^2+1}\right)+b\left(\frac{2y}{x^2+y^2+1}\right)+c\left(\frac{x^2+y-1}{x^2+y^2+1}\right)+d=0.$$

The stereographic projection of $P(x,y)$ onto the unit sphere centered at the origin is given by:

$$P(x,y)\to P'\left(\frac{2x}{x^2+y^2+1},\frac{2y}{x^2+y^2+1},\frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

Thus, point P lies on circle C if and only if P' lies on the plane q: $aX+bY+cZ+d=0$. The plane q cuts the sphere in the stereographic image C' of C, and the pole Q of the plane (i.e. the vertex of the cone where the tangents to the sphere at C' meet) has coordinate $(a/d, b/d, c/d)$ and homogeneous coordinate $Q(a,b,c,d)$. Therefore, given circle $C(a,b,c,d)$ in the plane, we can visualize its associated coordinate representative as the point $Q(a,b,c,d)$ in projective 3-space via its stereographic projection.

6. Conclusion

There are many other theorems in the geometry of circles that can be obtained using the methods we have described, and the methods extend immediately to the geometry of spheres in higher dimensions. These results can be found in the references. Alexander [1] discusses the three-dimensional case and includes applications to non-Euclidean geometry.

Recently these coordinates and linear algebra methods for circles have been used by Lester [12, 13] to characterize the injections on the set of spheres in $\mathbb{R}^n$ that preserve pairs of spheres that intersect at a prescribed, fixed angle. Middleditch, Stacey and Tor [14] developed a computer graphics algorithm to find the intersection points of circles and lines using a variant of the coordinates that we have described. Thus, these methods for circles continue to find both theoretical and practical application.

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Four hundred years ago...

Adriaen van Roomen [1561-1615], known as Adrianus Romanus, of the Netherlands, found the value of π correct to fifteen decimal places, using regular inscribed and circumscribed polygons having 2^{30} sides.

Howard Eves,
An Introduction to the History of Mathematics
5th Edition, Saunders College Publishing, 1983.

Extended Pascal Triangles

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The Pascal triangle, with its associated properties, binomial coefficients, and Fibonacci numbers, is surely among the most familiar mathematical objects. Its "extended" versions, which arise in a fairly natural way, have entries and properties that are the natural generalizations of the original, and although not commonly known, are in many ways equally useful and interesting. In what follows we introduce these extended triangles and discuss a few of their properties and applications. The extended Pascal triangle, T_m, is the (left-justified) array of coefficients in the expansion of $(1+x+x^2+\cdots+x^{m-1})^n$, for $m, n \geqq 0$.

These arrays may have been first explicitly discussed by J. E. Freund in a 1956 paper [**1**], where they arise in the solution of a restricted occupancy problem. There are no references to any previous similar development in that paper, and no connection with a generating function. N. Ya. Vilenkin's 1971 book on combinatorics [**2**, Chap. 5] discusses these generalizations (there called m-arithmetical triangles), in this case arriving at them through chessboard problems; again, they are not connected with a generating function, and there are no references to previous work. S. J. Turner introduced these triangles (there called Pascal-T triangles, a name now also frequently used) in a 1979 paper [**3**] on a probability problem. In 1984 [**4**] and 1986 [**5**] the author proved a number of theorems on the counting properties of these arrays and gave examples of their use in combinatorics and reliability theory. Some problems related to the kinds of results discussed in this paper recently appear (although without an explicit connection to these triangles) in a collection [**6**] of problems on combinatorics by I. Tomescu (see, e.g., such problems as 1.9, 1.10, 1.11, 1.19).

1. The Extended Pascal Triangle T_m

In this section, we introduce the definition of the *extended Pascal triangle* and a few of its consequences and follow this with some remarks on how the notion of an "extended" triangle might arise by analogy with some properties of the ordinary Pascal triangle.

By analogy with the $C(n,k)$ notation for the binomial coefficients, we will use the notation $C_m(n,k)$ for the coefficient of x^k in $(1+x+x^2+\cdots+x^{m-1})^n$. That is, T_m is that array which has in row n and column k the number $C_m(n,k)$ defined, for $m, n, k \geqq 0$, by

$$(1+x+x^2+\cdots+x^{m-1})^n = \sum_{k=0}^{(m-1)n} C_m(n,k)x^k, \tag{1}$$

where these entries satisfy the recurrence relations

$$C_m(0,k) = \begin{cases} 1, & \text{for } k=0 \\ 0, & \text{for } k \geqq 1, \end{cases}$$

and

$$C_m(n,k) = \sum_{j=0}^{m-1} C_m(n-1,k-j), \quad \text{for } n \geqq 1. \tag{2}$$

(The relation (2) is proved by a straightforward induction on n (m is fixed): Write out (1) in terms of $n-1$,, multiply both sides by $(1+x+\cdots+x^{m-1})$, and collect coefficients of powers of x on the right.)

It follows then that T_0 is the all-zero array except for $C_0(0,0)=1$, and that all the rows of T_1 consist of a one followed by zeros. For $m \geqq 2$, T_m is the array whose $n=0$ row is a one followed by zeros, whose $n=1$ row is m ones followed by zeros, and any of whose entries in subsequent rows is the sum of the m entries just above and to the left (with zeros making up any shortages near the left hand edge). The first few rows, for example, of T_5, showing the entries $C_5(n,k)$ are shown in TABLE 1.

TABLE 1.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	0	0	0	0	0
2	1	2	3	4	5	4	3	2	1	0
3	1	3	6	10	15	18	19	18	15	10
4	1	4	10	20	35	52	68	80	85	80
5	1	5	15	35	70	121	185	255	320	365
6	1	6	21	56	126	246	426	666	951	1246

We readily verify, for instance, that $C_5(3,5)=18$ is the sum $(2+3+4+5+4)$ of the five entries $C_5(2,1)$, $C_5(2,2),\ldots,C_5(2,5)$ just above and to the left.

As to the reasons for framing the definition of T_m in just the way it was given, and its connections with properties of the usual Pascal triangle, we offer the following remarks. If we assume, as seems reasonable, some standard properties of the usual Pascal triangle, here are some ways in which we might be led to the notion of an "extended" triangle. We know, for example, that we can consider the Pascal triangle (in its left-justified form) to be generated entirely from its first row (1 0 0...) by the rule that each entry in subsequent rows should be the sum of the *two* elements just above and to the left, with zeros making up any shortages along the left edge. Suppose we keep the same first row but require that elements in following rows be the sums of the *three* (or *four*, or *five*, ...) elements just above and to the left. This will produce, for a three-rule, the array (which we may as well call T_3)

T_3:

$n \backslash k$	0	1	2	3	4	5	6	7	
0	1	0	0	0	0	0	0	0	...
1	1	1	1	0	0	0	0	0	
2	1	2	3	2	1	0	0	0	
3	1	3	6	7	6	3	1	0	

...

and similarly for a four-rule, we have T_4 and so on. What to do in the case of an m-rule, then, is pretty clear.

Now, we also know that the nonzero entries in successive rows in the Pascal triangle are the coefficients in the expansions of $(1+x)^0$, $(1+x)^1$, $(1+x)^2,\ldots$. By analogy, then, the three ones in the second row of the three-array might be the coefficients in the expression $(1+x+x^2)$, and those in the second row of the

	$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	
T_4:	0	1	0	0	0	0	0	0	0	0	0	0	...
	1	1	1	1	1	0	0	0	0	0	0	0	
	2	1	2	3	4	3	2	1	0	0	0	0	
	3	1	3	6	10	12	12	10	6	3	1	0	

...

four-array suggest the coefficients in $(1+x+x^2+x^3)$. A little experimentation with powers of $(1+x+x^2)$ does in fact indicate that the numbers in the three-array are the coefficients in the expansion of $(1+x+x^2)^n$, for $n=0,1,2,\ldots$, and similarly for the four-array.

We probably see enough by now to say how we want things to be in the general case: The definition of T_m should incorporate these aspects of the Pascal triangle, but also extend them in as natural a way as possible, and this is what has been done. To see whether, or how, the extension may be in some degree a "success", we need now to look at what follows from it.

2. Some Elementary Properties of T_m

a) Let's observe at the outset that in the notation we have adopted, T_2 is of course *the* Pascal triangle, and $C_2(n,k)$ is the binomial coefficient $\binom{n}{k}$. It is also true that the generating function associated with T_m, namely $(1+x+\cdots+x^{m-1})^n$, can be written as $(1-x^m)^n/(1-x)^n$; the numerator can be expanded by the binomial theorem and the denominator by the binomial series, and collecting the coefficients of x^k gives a formula for $C_m(n,k)$ entirely in terms of binomial coefficients:

$$C_m(n,k)=\sum_j(-1)^j\binom{n}{j}\binom{n-1+k-mj}{n-1}. \tag{3}$$

In this discussion, however, we will prefer to think of $C_m(n,k)$ as belonging to its "own" triangle T_m, and for many purposes to regard T_2 pretty much as just another triangle in this infinite collection.

b) The row entries in the portions of T_3, T_4, T_5 given as examples seem to exhibit the same kind of symmetry as do the ordinary binomial coefficients of T_2. This is actually the case for all the T_m, and in general: For $m>0$ there are $((m-1)n+1)$ nonzero entries in row n of T_m, and the symmetry relation among them is

$$C_m(n,k)=C_m(n,(m-1)n-k), \qquad 0\leqq k\leqq(m-1)n. \tag{4}$$

For $m=2$, this is the well-known relation among the binomial coefficients. We can argue that (4) is true in general by observing that equation (1) shows that $C_m(n,k)$ may be also interpreted as the number of ways of distributing k like objects among n cells, where at most $(m-1)$ objects per cell are allowed. (This is because in every occurrence of x^k in the expansion of $(1+x+\cdots+x^{m-1})^n$, the exponent k is the sum of n summands whose values are between 0 and $(m-1)$.) But every such distribution uses only k of the $(m-1)n$ possible places, and so with that distribution we can associate another distribution of $(m-1)n-k$ objects among the n cells. That is, the sets counted by the left and right sides of (4) are equinumerous.

c) There is also a relation between the entries in "adjacent" triangles T_m and T_{m-1}, for $m \geqq 2$, which again comes from (1). Suppose we group the sum $1 + x + \cdots + x^{m-1}$ in parentheses in (1) as $[1 + (x + x^2 + \cdots + x^{m-1})]$ and expand the nth power of this binomial arrangement; the result is

$$\sum_{k=0}^{(m-1)n} C_m(n,k)x^k = \sum_{j=0}^{n} \binom{n}{j} x^j (1 + x + \cdots + x^{m-2})^j.$$

Now use (1) to replace $(1 + x + \cdots + x^{m-2})^j$ by its equivalent in terms of C_{m-1}'s, collect the coefficients of x^k on the right, and equate to the coefficient of x^k on the left. We find that

$$C_m(n,k) = \sum_{j=0}^{n} \binom{n}{j} C_{m-1}(j, k-j). \tag{5}$$

The array for T_5 tells us, for example, that $C_5(3,5) = 18$. Equation (5) tells us that $C_5(3,5)$ is also given by

$$\begin{aligned} C_5(3,5) &= \sum_{j=0}^{3} \binom{3}{j} C_4(j, 5-j) \\ &= 1 \cdot C_4(0,5) + 3 \cdot C_4(1,4) + 3 \cdot C_4(2,3) + 1 \cdot C_4(3,2) \\ &= 1 \cdot 0 + 3 \cdot 0 + 3 \cdot 4 + 1 \cdot 6 = 18. \end{aligned}$$

Repeated use of (5) would allow $C_m(n,k)$ to be expressed entirely in terms of binomial coefficients, somewhat as in (3). But (3) and (5) also perhaps indicate why, except for $m = 2$, there is not in general a simple explicit formula for $C_m(n,k)$ analogous to $C_2(n,k) = \binom{n}{k}$.

d) This is not an exhaustive list of properties of T_m, but perhaps one other aspect might be of interest here. We observe, as in [4], that the numbers that are the successive southwest-northeast diagonal sums of the elements of T_2 (starting with the $(1,0)$ entry) give a slightly modified version of the Fibonacci sequence (omitting the usual leading zero and one). That is, we get the recurrence (indexed by $n \geqq 0$, and denoted here by $f_2(n)$)

$$\begin{cases} f_2(0) = 1, \quad f_2(1) = 2 \\ f_2(n) = f_2(n-1) + f_2(n-2), \quad n \geqq 2, \end{cases}$$

which generates the familiar $1, 2, 3, 5, 8, \ldots$. (The reason is that any element in any diagonal is itself a sum of two elements, in the two preceding diagonals; it follows that the diagonal sums themselves satisfy the recurrence.) The applications of the Fibonacci sequence are many and varied, but we might mention just one uncommon aspect here as an example: $f_2(n)$ enumerates the number of binary strings of length n that do not contain two (or more) consecutive 1's. (This is because the members of such a collection end either with 0, and these are enumerated by $f_2(n-1)$, or with 01, and these are enumerated by $f_2(n-2)$. Thus the recurrence $f_2(n) = f_2(n-1) + f_2(n-2)$, with the same initial conditions.)

Suppose now we try the same thing with any triangle T_m. That is, take the sequence $\{f_m(n)\}_{n=0}^{\infty}$, $m, n \geqq 0$, to be the sequence of diagonal sums in T_m

$$f_m(n) = \sum_{j=0}^{n} C_m(n-j+1, j). \tag{6}$$

Then it will follow (since each C_m is a sum of m previous elements) that $f_0(n) = 0$ and $f_1(n) = 1$ for all n, and for $m \geqq 2$,

$$f_m(n) = \begin{cases} 2^n, & 0 \leqq n \leqq m-1 \\ \sum_{i=n-m}^{n-1} f_m(i), & n \geqq m. \end{cases} \tag{7}$$

We might then call (7) the *Fibonacci sequence of order m*, in which each element after the mth is the sum of the preceding m elements. Then $\{f_2(n)\}$ will be the usual Fibonacci sequence referred to above, but $\{f_3(n)\}$ will be $1, 2, 4, 7, 13, 24, 44, \ldots,$ and so on. And to continue the example (as the reader might show by an argument similar to that given for $f_2(n)$), $f_m(n)$ enumerates the number of binary strings of length n that do not contain m (or more) consecutive 1's.

3. T_m as the Natural Setting for Some Enumeration Problems

Consider the following counting problems: Enumerate the number (of ways) $W(s, n, m)$

a) that a given sum, s, can be thrown with n fair m-sided dice;
b) of solving the equation $x_1 + x_2 + \cdots + x_n = s$ in positive integers not exceeding a given integer m;
c) of compositions (ordered partitions) of s into exactly n positive parts with no part greater than m;
d) that s identical objects can be placed in n cells with each cell containing at least one object and at most m objects.

These are of course equivalent to each other, and to other similar formulations. If we consider that the generating function of the numbers $W(s, n, m)$ is

$$\phi(s, n, m, t) = \left(\sum_{i=1}^{m} t^i\right)^n = t^n \frac{(1-t^m)^n}{(1-t)^n},$$

we could choose to expand the last expression and evaluate $W(s, m, n)$ as the sum of products of binomial coefficients, as in [7],

$$W(s, n, m) = \sum_{k=0}^{U} (-1)^k \binom{n}{k} \binom{s-mk-1}{n-1}, \qquad U = [(s-n)/m], \tag{8}$$

where [ ] is the greatest integer notation.

Our intent here, however, is to point out that a "natural" setting for these kinds of problems is the extended Pascal triangle T_m (the computation of whose values even for large values of the parameters offers no essential difficulties). By "natural" we mean that 1) the evaluation of $W(s, n, m)$ is given by a direct one-term table-look-up in T_m, and 2) the interpretation of this way of evaluating $W(s, n, m)$ seems perspicuous in a way that (8) is not.

That is, if we write $\phi(s, n, m, t)$ by using (1) as

$$\begin{aligned} \phi(s, n, m, t) &= t^n(1 + t + \cdots + t^{m-1})^n \\ &= t^n \sum_{k=0}^{(m-1)n} C_m(n, k) t^k = \sum_{k=0}^{(m-1)n} C_m(n, k) t^{n+k}, \end{aligned}$$

then the coefficient of t^s is immediately available and so the resulting "direct" formula is

$$W(s,n,m) = C_m(n, s-n), \quad n \leqq s \leqq mn. \tag{9}$$

For example, the number of ways of obtaining the sum $s = 26$ with $n = 6$ ordinary ($m = 6$) dice is, by (9),

$$W(26,6,6) = C_6(6,20) = 2247.$$

Or, better still, because of the symmetry relation (4) we need only "half" of T_6 and so $W(26,6,6) = C_6(6,20) = C_6(6,10) = 2247$.

If the integral solutions of $x_1 + x_2 + \cdots + x_n = s$ are allowed to be nonnegative rather than positive, then of course the situation is even simpler, since the generating function is then $(1 + t + \cdots + t^m)^n = \sum_{k=0}^{mn} C_{m+1}(n,k)t^k$, and the coefficient of t^s is just $C_{m+1}(n,s)$. More generally, if the x_i are restricted to $0 \leqq p \leqq x_i \leqq q$ (and the generating function is $(t^p + t^{p+1} + \cdots + t^q)^n$), or when there are to be, say, n odd parts each $\leqq 2m+1$ (and the generating function is $(t + t^3 + t^5 + \cdots + t^{2m+1})^n$) then there is little change in difficulty. In the first case, for example, we would write

$$\begin{aligned}(t^p + t^{p+1} + \cdots + t^q)^n &= t^{np}(1 + t + t^2 + \cdots + t^{q-p})^n \\ &= t^{np} \sum_{k=0}^{(q-p)n} C_{q-p+1}(n,k)t^k \\ &= \sum_{k=0}^{(q-p)n} C_{q-p+1}(n,k)t^{k+np}.\end{aligned}$$

Then the coefficient of t^s will just be $C_{q-p+1}(n, s-np)$, whereas to do this in terms of ordinary binomial coefficients, this number has to be obtained by writing the generating function as $t^{np}(1 - t^{q-p+1})/(1-t)^n$, expanding the numerator and denominator, and collecting coefficients of powers of t, an elementary but not especially attractive calculation. We need of course to have a table of the C_{q-p+1}'s, but this is not difficult to obtain; in terms of simplicity, the idea of the natural setting seems fairly convincing.

4. Other Aspects of T_m

The preceding sections have introduced the extended Pascal triangles and have given some of their straightforward properties and applications. Lastly, here are two other quite different aspects of T_m in which open problems remain.

1) In [7], H. B. Mann and D. Shanks gave a novel criterion for primality in terms of displaced entries in T_2; the simple description is as follows. We displace the entries in each row two places to the right from the previous row (so that the $n+1$ entries in row n now start in column $2n$). Also, we circle the entries in row n that are divisible by n. Then the column number k is a prime if and only if all the entries in column k are circled. Most of us remember or can easily work out several rows of T_2, enough to try out those directions and see the results for the familiar triangle.

Suppose now we apply the same directions to the next simplest case, T_3. The first few rows of the result are shown in TABLE 2. It looks as though the criterion also works in T_3, and in fact it does, as shown in [9]. A little experimentation with $T_4, T_5, \ldots$, suggests the perhaps surprising conclusion that the criterion may apply in all triangles T_m, but this has not been proved for $m > 3$ (see Postscript). This is

TABLE 2 Primality criterion for T_3

	0	1	(2)	(3)	4	(5)	6	(7)	8	9	10	(11)	12	(13)	14	15	16	(17)
0	1																	
1			(1)	(1)	(1)													
2					1	(2)	3	(2)	1									
3							1	(3)	(6)	7	(6)	(3)	1					
4									1	(4)	10	(16)	19	(16)	10	(4)	1	
5											1	(5)	(15)	(30)	(45)	51	(45)	(30)
6													1	(6)	21	50	(90)	(126)
7															1	(7)	(28)	(77)
8																	1	(8)

perhaps not a problem with momentous consequences, but is nevertheless an interesting one.

2) Suppose we reduce the entries in T_2 modulo 2; we get, in part, the array in FIGURE 1.

The reader can continue this (or look, for example, at the figure in **[10]**, which is a one-page paper consisting essentially of a continuation of this picture through row $n = 31$), but already we see that there seem to be some patterns forming. We might ask, for instance, how many nonzero entries there are in row n; what are the maximal and minimal numbers of these nonzero entries, and in what rows do they occur? Is there a formula for the number of values of k for which $C_2(n, k) \not\equiv 0 \pmod 2$? What about $C_2(n, k) \not\equiv 0 \pmod p$?

The answers for T_2 are known for any prime p (**[11]**, results and references, sections 6, 7).

a) If p is a prime, and $N_2(n, p)$ is the number of values of k for which $C_2(n, k) \not\equiv 0 \pmod p$, and if $n = n_0 + n_1 p + \cdots + n_r p^r$, then
$$N_2(n, p) = \prod_{i=0}^{r} (n_i + 1);$$
b) $N_2(n, p) \geqq 2$, with equality holding if and only if $n = p^s$, where s is a nonnegative integer;

c) $N_2(n, p) \leqq n + 1$; equality holds if and only if $n = ap^s - 1$, with $1 \leqq a \leqq p$.

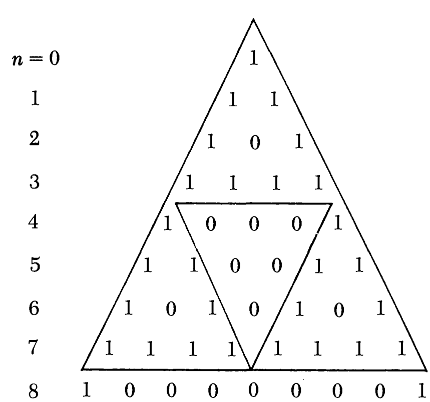

FIGURE 1

That is, all we need (according to (a)) are the digits in the p-ary representation of n. Of course things are simplest of all when $p = 2$: $N_2(n, 2)$ is just 2 raised to the number of 1's in the binary representation of n.

As for the corresponding questions for T_m, $m > 2$, there are some partial answers. The author and C. L. Burchard show in [**12**], for example, that if we reduce the entries of T_p modulo p, what is true is that:

d) If p is a prime, and $N_p(n, p)$ is the number of values of k for which $C_p(n, k) \not\equiv 0 \pmod p$, and if $(p-1)n = a_0 + a_1 p + \cdots + a_r p^r$, then

$$N_p(n,p) = \prod_{i=0}^{r} (a_i + 1).$$

e) $N_p(n, p) \geqq p$; $N_p(n, p) = p$ if and only if $n = p^s$, where s is an integer $\geqq 0$;

f) $N_p(n, p) \leqq (p-1)n + 1$; $N_p(n, p) = (p-1)n + 1$ if and only if

$$n = (p^s - 1)/(p - 1).$$

Similar statements are true [**12**] when m is a power of a prime p and the reduction is modulo p, but the question is still open as to what might be the case for m composite (and not a power of a prime). We do not know the answer in the case of $T_p \pmod q$, where p and q are different primes. These problems for extended Pascal triangles are thus related to the many results in the literature on the divisibility of binomial and multinomial coefficients by a prime or prime power.

In conclusion, we hope that the discussion has shown that the T_m arrays really are "extensions" of the Pascal triangle, with many similar properties that seem to be the natural generalizations of those of T_2, but perhaps with a few surprises also. T_2 has certainly been a rich source of interesting and useful mathematics. We suggest that its extended relatives potentially may serve as equally fruitful objects of study.

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Postscript. The conjecture is indeed true; the criterion applies in all triangles T_m. See W. Adams, D. Shanks, and E. Liverance, Infinitely many necessary and sufficient conditions for primality, *Bull. Inst. of Combinatorics and Its Appl.*, Vol. 3 (1991) 69–76.

A Set is a Set (Sung to the tune of the theme to "Mr. Ed")

A set is a set.
(You bet! You bet!)
And nothing could not be a set,
You bet!
That is, my pet
Until you've met
My very special set!

If this were a set,
It'd be a threat,
And lead to conclusions
That you'd regret.
And make you fret
And wet with sweat—
This very special set!

Let *A* be the set of every *U*
That doesn't belong to *U*.
Then if *A*'s in *A*, it's not in *A*
And if not, then what can you do?

So don't use the het-
Erological set
'Cause some things cannot
Be a set, my pet.
Or, better yet,
Go out and get
The *class* of every set.

Oh, Bertrand.

—Bruce Reznick
University of Illinois
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NOTES

Graceful Configurations in the Plane

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1. Introduction Let C be a finite configuration of (infinite) lines in the plane. The lines of C partition the plane into regions $R_0, R_1, \dots, R_m$. Suppose it is possible to label these regions with all the integers $\{0, 1, \dots, m\}$, say R_k is labeled with $\lambda(R_k)$, so that if R_i and R_j share a common boundary line L of C, then $\lambda(R_i) - \lambda(R_j)$ only depends on L. In this case we say that λ is a *graceful* labeling of C, and that C is a *graceful* configuration. In FIGURE 1, we show a variety of graceful configurations.

The concept of a graceful configuration was introduced by D. E. Knuth [3], who also raised the following general question: What are the graceful configurations? In this note we explore this question. In particular, we describe several infinite families of graceful configurations (Sec. 3), as well as several infinite families of nongraceful configurations (Sec. 4, 5). We conclude with a number of open problems. This topic can be viewed as a type of geometrical analogue to some well-studied questions in graph theory (cf. [1], [2]).

2. Preliminaries The reader may notice that the two labelings of the same configuration C of four lines shown in FIGURE 1(e) and (f) differ in the following way. In (f), when crossing the line L from left to right, the region values change both by -1 (from 3 to 2) and by $+1$ (from 4 to 5). However, in (e), as in all the other labelings in FIGURE 1 (except for (f)), the label differences are constant as we cross *oriented* lines of the configurations in the same direction. We call such labelings *strict* graceful labelings. Graceful labelings such as that in (f) will be called *twisted*. At present no configuration C is known that has a twisted graceful labeling but no nontwisted (i.e., ordinary) one. However, any such C must be rather special, as the following result shows.

THEOREM 1. *Suppose C has a graceful labeling in which the labeling of regions bordering the line L is twisted at a point p. Then p must lie on at least four lines of C.*

Proof. Suppose p lies on just two lines L and L'. Consider a twisted labeling shown in FIGURE 2(a).

Since the labeling is graceful we must have:

$$|a + k - b| = |a - (b + k)|.$$

There are two possibilities:

(i) $a + k - b = a - b - k \Rightarrow k = 0$ ~~X~~ (contradiction)

(ii) $a + k - b = -a + b + k \Rightarrow a = b$ ~~X~~

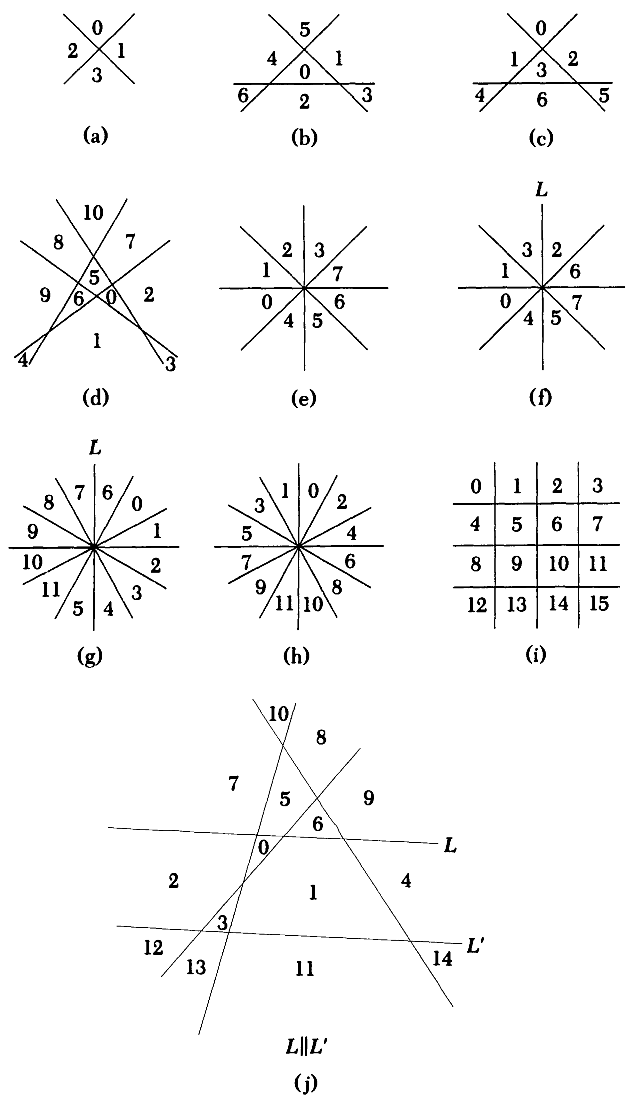


FIGURE 1

Some graceful configurations.

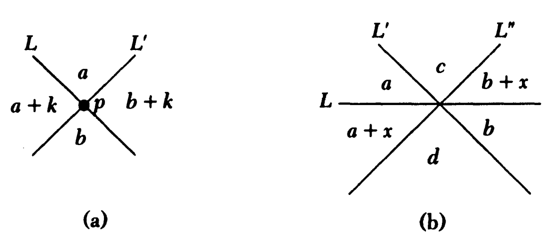


FIGURE 2

Now, suppose p lies on just three lines, L, L', and L''. Consider the twisted labeling shown in FIGURE 2(b). Again, since the labeling is graceful, we have:

$$|c-a| = |b-d|, \quad |b+x-c| = |a+x-d|.$$

There are now four possibilities, each leading to a contradiction.

(i)
$$\begin{array}{c} c-a=b-d \\ b+x-c=a+x-d \end{array} \Rightarrow \begin{array}{c} b-c=d-a \\ b-c=a-d \end{array} \Rightarrow a=d \quad \text{X}$$

(ii)
$$\begin{array}{c} c-a=b-d \\ b+x-c=d-a-x \end{array} \Rightarrow \begin{array}{c} b-c=d-a \\ x=-x \end{array} \Rightarrow x=0 \quad \text{X}$$

(iii)
$$\begin{array}{c} c-a=d-b \\ b+x-c=a+x-d \end{array} \Rightarrow \begin{array}{c} c-d=a-b \\ c-d=b-a \end{array} \Rightarrow a=b \quad \text{X}$$

(iv)
$$\begin{array}{c} c-a=d-b \\ b+x-c=d-a-x \end{array} \Rightarrow \begin{array}{c} c=a+d-b \\ c=a-d+b+2x \end{array} \Rightarrow c=a+x \quad \text{X}$$

This proves Theorem 1.

3. Some graceful configurations To begin with, all configurations with at most four lines are graceful. We list these in FIGURE 3.

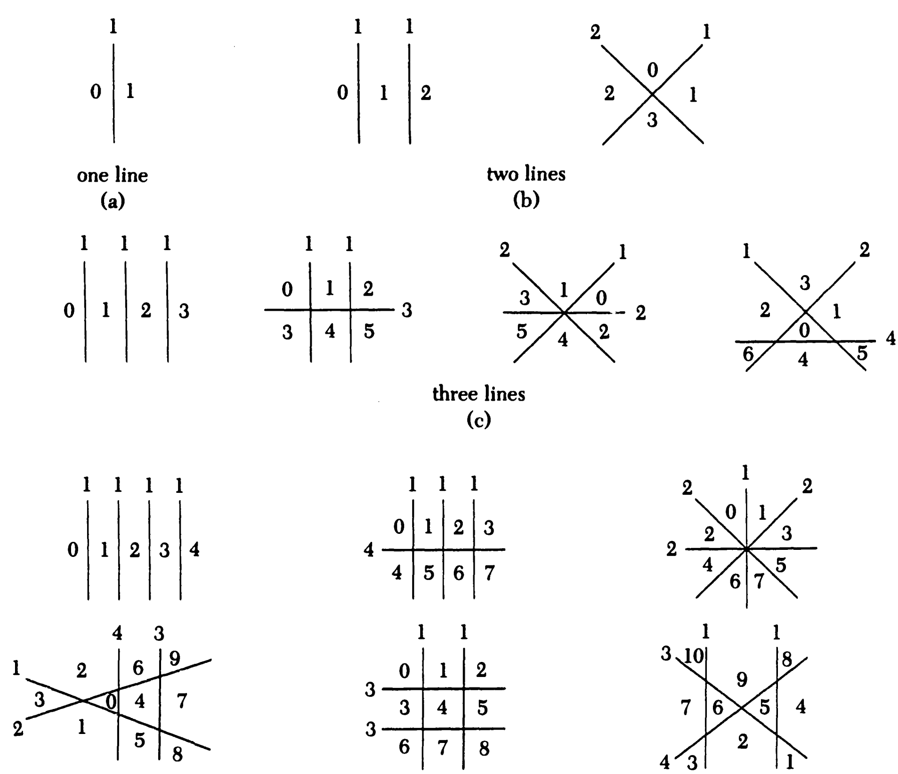
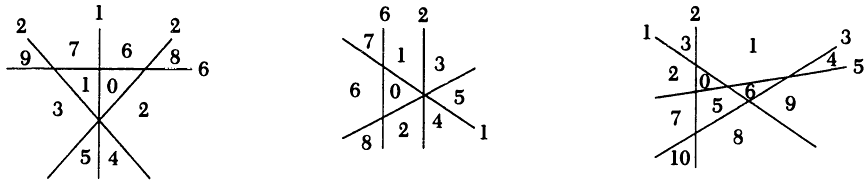


FIGURE 3



Four lines
(d)

FIGURE 3
(Continued).
Configurations with at most four lines.

Below we list several infinite families of graceful configurations.

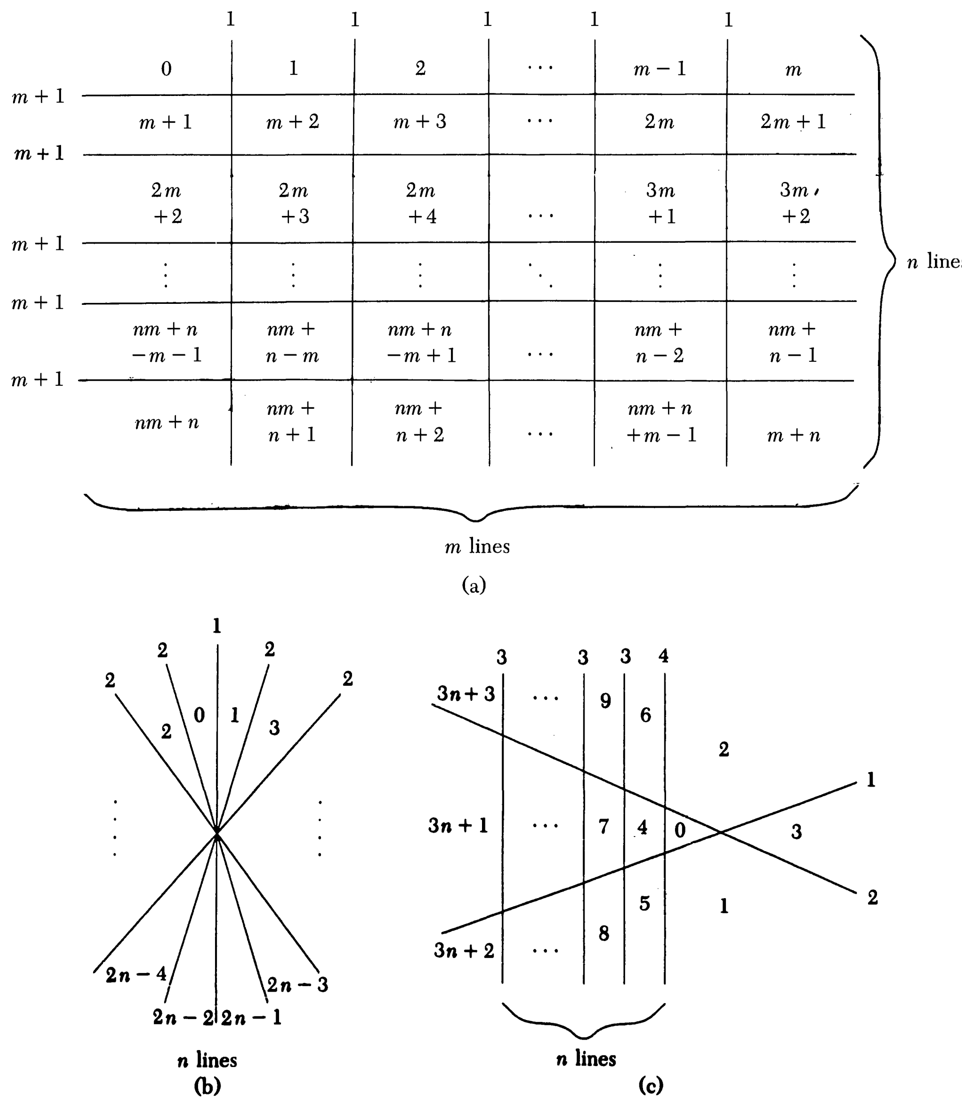
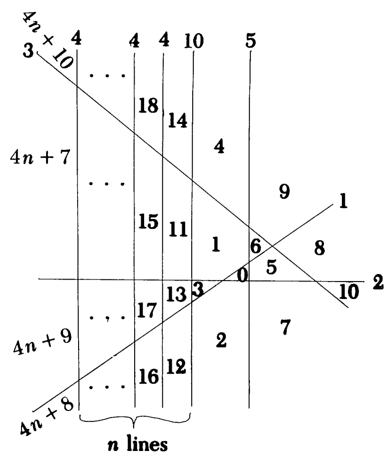
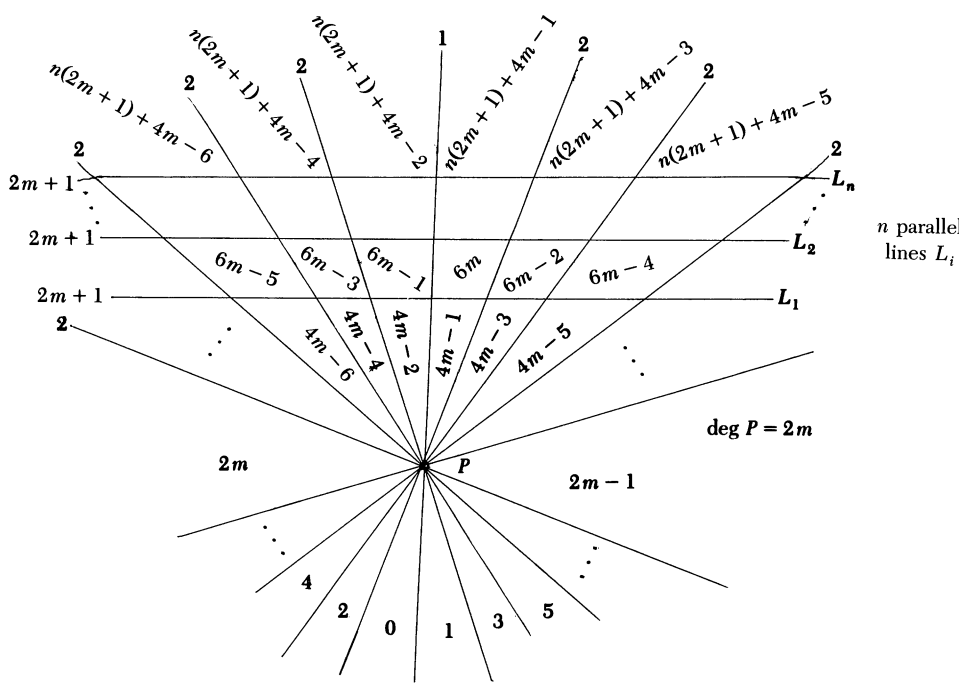


FIGURE 4
Some infinite families of graceful configurations.



(d)



(e)

FIGURE 4
(*Continued*).

4. The smallest nongraceful configuration Suppose the configuration C_5 of five lines shown in FIGURE 5 has a graceful labeling. By subtracting an appropriate value from each region we can assume that the central region is assigned the value 0, where the region values now form an *interval* $x, x+1, \ldots, x+15$ with x a non-positive integer. If we assign the values a_i to the five regions adjacent to the central region (see FIGURE 5) then the other 10-region values are as shown. Define S to be the sum of the 16-region values. Thus,

$$S = 6\sum_i a_i = \sum_{j=0}^{15} (x+j) = 16x + 120. \tag{1}$$

Therefore, 3 divides x, say $x = 3y$ and

$$\sum_i a_i = 8y + 20. \tag{2}$$

Note that replacing each a_i by $-a_i$ if necessary, we can assume that $S \geq 0$, i.e., $y \geq -2$. Because 0 is a region value, we must have $y \leq 0$. Thus, there are three possibilities: $y = -2, -1$, and 0. However, each of these three cases can be ruled out by straightforward case enumeration, and we conclude by Theorem 1 that C_5 is not graceful.

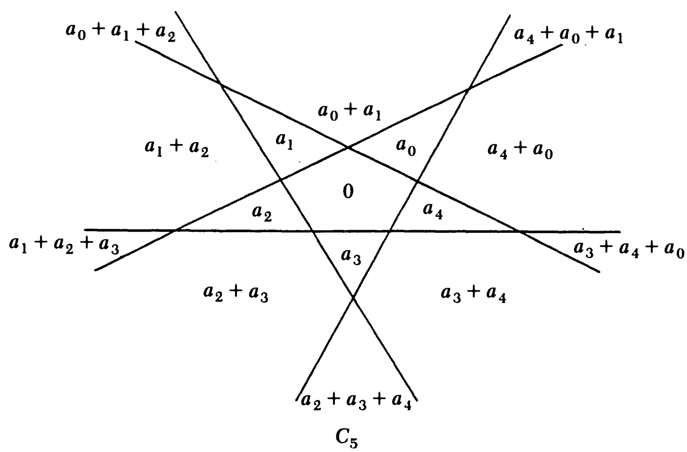


FIGURE 5

5. More families of nongraceful graphs It is possible to generalize the construction of C_5 in the preceding section to the more general configurations C_{2r+1} formed by extending the edges of a regular $(2r+1)$-gon. As before, we assume that C_{2r+1} has been gracefully labeled and we normalize the region values so that the central region has the value 0, with the adjacent regions having the values a_i, $0 \leq i \leq 2r$. An easy calculation shows that the total number R of regions is $1 + (r+1)(2r+1)$. We denote the resulting interval of (normalized) region values by

$$x, x+1, \ldots, x + (r+1)(2r+1).$$

It is not difficult to verify (similar to the previous argument for C_5) that the region values are exactly all the sums $a_i + a_{i+1} + \cdots + a_{i+k}$ for $0 \leq k \leq r$, together with 0, where the index addition is performed modulo $2r+1$.

We next express the sum S of the region values in two ways. On one hand,

$$S = \sum_{j=0}^{R-1} (x+j) = Rx + \binom{R}{2}. \tag{3}$$

On the other,

$$S = \binom{r+2}{2} \sum_{i=0}^{2r} a_i \tag{4}$$

since each a_i occurs exactly $\binom{r+2}{2}$ times. Now, suppose $r = 8t + 6$. Then by (3) and (4),

$$\begin{aligned}\sum_{i=0}^{r} a_i &= \frac{Rx + \binom{R}{2}}{\binom{r+2}{2}} \\ &= \frac{R(2x+R-1)}{(r+2)(r+1)} \\ &= \frac{(1+(r+1)(2r+1))(2x+(r+1)(2r+1))}{(r+2)(r+1)} \qquad (5) \\ &= \frac{(64t^2+108t+46)(2x+(8t+7)(16t+13))}{(8t+7)(8t+8)} \\ &= \frac{(32t^2+54t+23)(2x+(8t+7)(16t+13))}{2(8t+7)(t+1)}.\end{aligned}$$

However, this is clearly impossible since the numerator is odd, the denominator is even and $\sum_{i=0}^{r} a_i$ is an integer. This proves

THEOREM 2. *C_{16t+13} is not graceful for $t \geq 0$.*

We conclude this section with several doubly infinite classes of configurations that are not strictly graceful.

Define $C_n^{(m)}$ to be the configuration formed by starting with the "extended edges of a regular n-gon" configuration C_n and replacing each of the n lines with very closely spaced m parallel lines. We now modify $C_n^{(m)}$ by moving the m lines in each parallel class c_i so as to go through a single point p_i. The points p_i are chosen symmetrically around the center of the configuration, and very far away from it. The resulting configuration of mn lines, having rotational symmetry of $2\pi/n$ radians, we denote by $\overline{C}_n^{(m)}$. We show a portion of $\overline{C}_n^{(m)}$ in FIGURE 6.

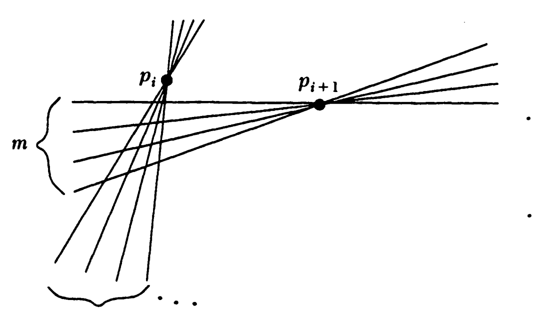


FIGURE 6
A portion of $\overline{C}_n^m$.

In FIGURE 7 we show a portion of $\overline{C}_n^{(m)}$ with values assigned to the regions, normalized from a strictly graceful labeling so that the central region has value 0.

We assume from now on that $n = 2t + 1$ is odd. It is not hard to check that the total number $\overline{R}$ of regions in $\overline{C}_{2t+1}^{(m)}$ is

$$\overline{R} = 1 + (2t+1)(tm^2 + 2m - 1). \qquad (6)$$

Thus, the assumption that $\overline{C}_{2t+1}^{(m)}$ is (strictly) gracefully labeled, implies that for some x, the region values are $x, x+1, x+2, \dots, x+\overline{R}-1$.

As before, we now compute the sum $\overline{S}$ of all the region values in two ways. On one hand,

$$\overline{S} = \sum_{j=0}^{\overline{R}-1} (x+j) = \overline{R}x + \binom{\overline{R}}{2}. \tag{7}$$

On the other hand, each $a_i(j)$, $0 \le i < 2t+1$ occurs equally often, say $R(j)$ times, in the region values (by symmetry), where

$$R(j) = \frac{t(t+3)}{2}m^2 + (t+1)m - t - (j-1)2tm$$

by a straightforward (but perilous) computation. Thus,

$$\overline{S} = \sum_{i=0}^{2t} \sum_{j=1}^{m} R(j)a_i(j). \tag{8}$$

Our final step is to make certain modular assumptions on m and $n = 2t+1$ to obtain a contradiction, thereby showing that for these m and n, no strictly graceful labeling of $\overline{C}_n^{(m)}$ exists.

For the first choice, we take:

$$t = 4u, \qquad m = 4v,$$

with u and v odd, and $u - v \equiv 4 \pmod 8$. Then $\overline{R}$ is even, and an easy computation shows that $\overline{S} \not\equiv 0 \pmod{32}$ but that $R(j) \equiv 0 \pmod{32}$ for all j. This clearly contradicts (8).

For the second choice, we take

$$t = 4u+2, \qquad m \equiv 2u+3 \pmod 4.$$

Then a similar calculation now shows that

$$\overline{S} \not\equiv 0 \pmod 4$$

but that $R(j) \equiv 0 \pmod 4$ for all j, again contradicting (8). Thus we have the

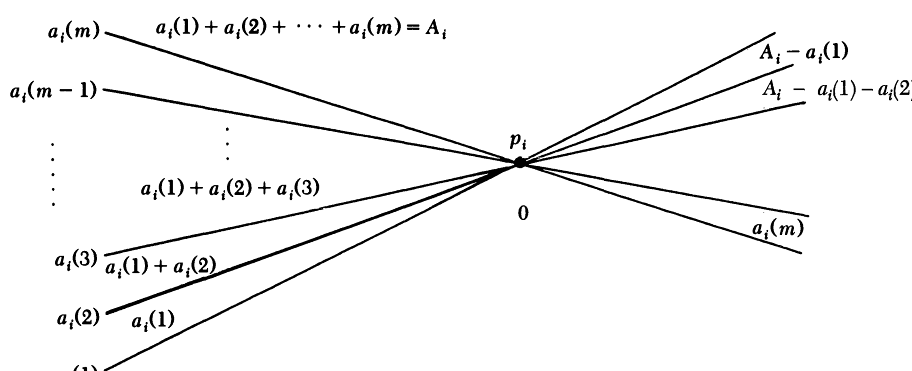


FIGURE 7
Generic labels assigned to regions of $\overline{C}_n^{(m)}$.

following theorem.

THEOREM 3. *If*

$$n = 8u + 5, \qquad m \equiv 2u + 3 \pmod 4)$$

or

$$n = 16w + 9, \qquad m \equiv 8w + 20 \pmod{32},$$

for nonnegative integers u and w, then $\overline{C}_n^{(m)}$ *is not strictly graceful.*

Since for $m \geq 4$, $\overline{C}_n^{(m)}$ has points lying on more than three lines, we cannot rule out the possibility of a twisted graceful labeling.

6. Concluding remarks A number of challenging open questions remain unanswered. We list several of these below.

(i) Is there *any* graceful configuration consisting of five or more lines in general position (i.e., no two parallel and no three concurrent)? We suspect that there are not. (The 5-line configuration shown in FIGURE 1(j) is a "near miss".) For example, are the extended edge configurations C_{2k+1} for regular $(2k + 1)$-gons all nongraceful?

(ii) An easier exercise would be to show that almost all configurations are not graceful (something we definitely believe). If there is a simple counting argument showing this, it has eluded us.

(iii) Are there configurations that have only *twisted* graceful labelings, or does the existence of a twisted labeling imply the existence of a nontwisted one? The configurations $\overline{C}_n^{(m)}$ in Section 5 with $m \geq 4$ might be good candidates for such configurations.

(iv) What are the analogous results (and questions!) in three or more dimensions?

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In the Gaussian Integers, $\alpha^4 + \beta^4 \neq \gamma^4$

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We recall the Fermat Conjecture: If n is an integer greater than 2, then there exists no triple $\{x, y, z\}$, of positive integers such that

$$x^n + y^n = z^n.$$

This simple statement was made by Fermat in the mid-1600s as a marginal note in his copy of the work of Diophantus and accompanied by the remark that he had found a marvelous proof that was too long to be recorded on the free space on the page. It has not ceased to tantalize amateur and professional alike since it was found after Fermat's death and made public by his son.

Since a proof for $n = 4$ was already known (due to Fermat himself) and since $x^{ab} + y^{ab} = z^{ab}$ implies that $(x^a)^b + (y^a)^b = (z^a)^b$, it follows that the truth of the conjecture for n an odd prime would guarantee its truth for all $n > 2$. Although proofs for some of the odd primes were forthcoming during the next two centuries, these proofs were frustratingly specific to particular values of n. Then in the mid-1800s new insight suggested that unity for the odd primes could be attained by viewing the set Z of rational (ordinary) integers as a subset of certain sets of complex numbers, which were called cyclotomic integers, and in which $x^n + y^n$ decomposes into n factors, all linear in x and y. We look below at these "integers," whose counterpart for $n = 4$ provides the setting for the theorem that we will prove. It appeared for a time that a strategy based on this new insight would finally settle the matter. It didn't, the spoiler being that, although cyclotomic integers share many properties with Z, including that of being expressible as products of primes, in some cases this factorization is not unique. (The reader who is not familiar with the fascinating history of the Fermat Conjecture is assured that there is an abundance of pertinent literature and is invited in particular to consult references [1], [3], [4], and [5], in which many other sources are suggested.)

We begin our introduction to cyclotomic integers by considering the special case, $n = 3$, of the conjecture. Let Q denote the field of rational numbers and let ω_3 denote $(-1 + \sqrt{-3})/2$, a primitive solution of the equation, $x^3 - 1 = 0$. We let $Q(\omega_3)$ denote the field "Q adjoin ω_3". The "cyclotomic integers" for $n = 3$ are the algebraic integers, I_3, of $Q(\omega_3)$:

$$Q(\omega_3) = \{a + b\omega_3 \colon a \text{ and } b \text{ are in } Q\}, \text{ while } I_3 = \{a + b\omega_3 \colon a \text{ and } b \text{ are in } Z\}.$$

Now it can be shown (for example, [4], pp. 169–176, or [5], pp. 191–194) that there is no triple, $\{\alpha, \beta, \gamma\}$, of nonzero members of I_3 such that $\alpha^3 + \beta^3 = \gamma^3$. Then, since Z is included in I_3, the Fermat Conjecture is true for $n = 3$.

More generally, let p denote an odd prime, let

$$\omega_p = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p},$$

a primitive solution of the equation, $x^p - 1 = 0$, and let $Q(\omega_p)$ denote Q adjoin ω_p. The "cyclotomic integers" for $n = p$ are the algebraic integers, I_p, of $Q(\omega_p)$:

$$Q(\omega_p) = \left\{a_0 + a_1\omega_p + a_2(\omega_p)^2 + \cdots + a_{p-2}(\omega_p)^{p-2} \colon \text{the } a_i\text{'s are in } Q\right\},$$

while

$$I_p = \left\{a_0 + a_1\omega_p + a_2(\omega_p)^2 + \cdots + a_{p-2}(\omega_p)^{p-2};\text{ the } a_i\text{'s are in Z}\right\}.$$

Again, we should note that Z is included in I_p and that it appeared for a time that the conjecture would be proven through study of the problem in I_p. In fact, Kummer and others ([3], p. 172) showed that for certain odd primes called regular primes, unique factorization could be restored in I_p in a sense sufficient to prove the conjecture valid not only for Z-integers but for the larger set of cyclotomic integers as well.

It is the case $n = 4$ that is of interest here. In elementary texts ([1], p. 291, for example), Fermat's proof by infinite descent is given to establish the conjecture easily for $n = 4$ without viewing the set of positive integers as a subset of I_4 (the "integers" of $Q(\omega_4)$, where $\omega_4 = i$ is a primitive solution of $x^4 - 1 = 0$). Reflection on this point led us to think that it would be pleasing to see the Z-proof of the conjecture for this case come as a corollary to an I_4-proof, and thus we are led to offer our argument that for $n = 4$ the conjecture is valid in I_4, which is, of course, the set of Gaussian integers. Our exposition is, with minimal assistance from their teachers, accessible to undergraduates.

The Gaussian integers For convenience we are going to denote I_4, the Gaussian integers, by G. In fact, G is the subset of the complex numbers consisting of all $x + yi$, where each of x and y is a rational integer. G is a unique factorization domain in which the units (members with multiplicative inverses) are 1, i, -1, and $-i$. If $\alpha = x + yi$ is in G, we define the norm of $\alpha, N(\alpha)$, to be $x^2 + y^2$ (a nonnegative integer). The norm of a product is the product of the norms, and $N(\alpha)$ divides $N(\beta)$ if α divides β.

We now give some pertinent facts about G that the reader can find discussed in more detail in [2]. The complex number $\delta = 1 + i$ is a prime in G (and a factor of 2) that plays a role in G rather like that played by 2 in the rational integers Z. If we let α in G be called "even" or "odd" according to whether or not α is divisible by δ, then the sum of two even members of G is even, the sum of two odd members is even, the sum of an even member and an odd member is odd, the product of two odd members is odd, and the product of an even member with any member is even. The member $x + yi$ of G is even if and only if $x \equiv y \pmod 2$.

If $\{\xi, \psi, \zeta\}$ is a triple of nonzero members of G with $\gcd(\xi, \psi) = 1$ and such that $\xi^2 + \psi^2 = \zeta^2$, then exactly one member of the triple is even. We define a Primitive Pythagorean Triple (PPT) in G to be an ordered triple (ξ, ψ, ζ) of nonzero members of G with $\gcd(\xi, \psi) = 1$, having ξ and ζ odd while ψ is even, and such that

$$\xi^2 + \psi^2 = \zeta^2.$$

(If $\gcd(\xi, \psi) = 1$, then the members of the triple are relatively prime in pairs and $\gcd(\xi, \psi, \zeta) = 1$.) Reminiscent of the familiar means by which Pythagorean Triples in the integers Z can be generated, PPTs in G can be generated by relatively prime odd Gaussian integers (see [2], p. 108). More precisely for our purposes, let (ξ, ψ, ζ) be a PPT in G; then there exist units E_1, E_2, and E_3 in G, and relatively prime odd members a and b of G, with real parts odd such that

$$E_1\xi = \frac{a^2 + b^2}{2}, \qquad E_2\psi = \frac{a^2 - b^2}{2i}, \qquad E_3\zeta = ab.$$

Example. The triple $(\xi, \psi, \zeta) = (-4+i, 4+8i, 4+7i)$ is a PPT. If we let $E_1 = E_2 = E_3 = -i$, while $a = 3+2i$ and $b = 1-2i$, we find that $(E_1\xi, E_2\psi, E_3\zeta) = (1+4i, 8-4i, 7-4i)$ is also a PPT and is generated by a and b in the sense that $1+4i = (a^2+b^2)/2$, $8-4i = (a^2-b^2)/2i$, and $7-4i = ab$. We point out that a and b are odd and have odd real parts.

What about $\alpha^4 + \beta^4 = \gamma^4$ in G? As we said earlier, we are going to show that there is no triple $\{\alpha, \beta, \gamma\}$ of nonzero members of G such that $\alpha^4 + \beta^4 = \gamma^4$. Then, since Z is a subset of G, it will follow that the Fermat Conjecture is valid for $n = 4$. Since $(\rho\alpha)^4 + (\rho\beta)^4 = (\rho\gamma)^4$ implies $\alpha^4 + \beta^4 = \gamma^4$, it suffices to prove that no such triple exists with $gcd(\alpha, \beta) = 1$. In fact, we show a bit more:

THEOREM. *There exists no triple $\{\alpha, \beta, \gamma\}$ of nonzero members of G with $gcd(\alpha, \beta) = 1$ such that $\pm\alpha^4 \pm \beta^4 = \pm\gamma^4$.*

Proof. (Our proof is a version of Fermat's method of infinite descent.) Suppose such a triple does exist. It is easy to see that exactly one of α, β, and γ is even. Then it is possible to rename and write

$$\pm\alpha^4 \pm \beta^4 = \gamma^2. \tag{1}$$

where α and γ are odd while β is even and $\gcd(\alpha, \beta) = 1$. In the set of all such triples $\{\alpha, \beta, \gamma\}$, there is one in which $N(\gamma)$ is minimal. We assume that the triple under consideration is one such. Now $\alpha^4 = (\alpha^2)^2$ and $-\alpha^4 = (i\alpha^2)^2$; similarly $\beta^4 = (\beta^2)^2$ and $-\beta^4 = (i\beta^2)^2$. Thus there exist units, u_1 and u_2, such that $(u_1\alpha^2, u_2\beta^2, \gamma)$ is a PPT. Then we let E_1, E_2, and E_3 be units and a and b be relatively prime odd Gaussian integers with odd real parts such that

$$E_1\alpha^2 = \frac{a^2+b^2}{2}, \qquad E_2\beta^2 = \frac{a^2-b^2}{2i}, \qquad E_3\gamma = ab. \tag{2}$$

(Here we have used the fact that the units in G form a multiplicative group so that, for example, any unit times $u_1\alpha^2$ can be written as $E_1\alpha^2$.) Now one can check that if α is an odd Gaussian integer, then α^2 has odd real part and (by considering a^2 and $b^2 (\bmod 4)$) that if each of a and b is odd with odd real part, then so is $(a^2+b^2)/2$. These observations imply that the unit E_1 is either 1 or -1. Now we divide the middle equality of (2) by $-2i$:

$$-\frac{E_2\beta^2}{2i} = -E_2\left(\frac{\beta}{\delta}\right)^2 = \frac{a^2-b^2}{4} = \frac{a-b}{2}\,\frac{a+b}{2}.$$

Each of a and b has odd real part and even imaginary part, so that $(a-b)/2$ and $(a+b)/2$ are members of G. Moreover they are relatively prime, for if π is a common prime factor, then it divides their sum and difference; that is, it divides each of a and b, which is impossible. Then since their product is a unit times a square, each of $(a-b)/2$ and $(a+b)/2$ is a unit times a square:

$$\frac{a-b}{2} = U_1 s^2 \qquad \text{and} \qquad \frac{a+b}{2} = U_2 t^2, \quad \text{where } U_1 \text{ and } U_2 \text{ are units.}$$

Any common divisor of s and t would divide $(a-b)/2$ and $(a+b)/2$; then s and t are relatively prime. Now

$$\left(\frac{a-b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2 = \left(U_1 s^2\right)^2 + \left(U_2 t^2\right)^2 = \frac{a^2+b^2}{2} = E_1\alpha^2 = \pm\alpha^2.$$

Then, multiplying through by -1, if necessary, and noting that the square of any unit is either 1 or -1, we can write

$$\pm s^4 \pm t^4 = \alpha^2. \tag{3}$$

Here α is odd, so that exactly one of s and t is even. Without loss of generality, we assume that s is odd. We compare (1) and (3) and observe that if we can show that $N(\alpha) < N(\gamma)$, then we will have proven our theorem. To this end we recall that $E_1\alpha^2 = (a^2 + b^2)/2$ and $E_3\gamma = ab$. Since the norm of a unit is 1, we have

$$N(\alpha^2) = \frac{N(a^2 + b^2)}{4} \qquad \text{and} \qquad N(\gamma^2) = N(a^2b^2) = N(a^2)N(b^2).$$

Hence $N(a^2 + b^2) \leqslant N(a^2) + N(b^2)$, which is less than 4 $N(a^2)N(b^2)$, since for positive integers m and n, $m + n < 4mn$. Then $N(\alpha^2) < N(\gamma^2)$, which implies that $N(\alpha) < N(\gamma)$. This contradiction completes the proof of the theorem.

A question for the reader Have we used the fact that factorization into primes in G is unique, and if so, then where?

Thanks. I am grateful to my student, Danny K. McIntyre, whose interest in this problem was an inspiration to me. I also wish to thank my colleague, Professor Clay Ross, for his reading of the manuscript and to thank the referees for their valuable suggestions.

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The Range of the Adjugate Map

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In elementary courses in linear algebra, the student is introduced to a matrix of cofactors called the *adjoint matrix*. The name "adjoint" is probably not the best, however, because the term is also used to describe the conjugate of the transpose of a matrix, something that is usually quite different. To overcome this difficulty, mathematicians have tried to settle on a term that would not disturb much of traditional symbolism. One such term that many have accepted is that of *adjugate*, c.f. [**1**, p. 257], [**2**, p. 135], [**3**, p. xii]. Thus one may still use such a familiar notation as adj A. In this paper, we shall use the term adjugate in preference to any of the other names known for the matrix.

The concept of adjugate usually comes up during a discussion of inverse matrices. It is observed that if A is an $n \times n$ complex matrix with inverse A^{-1}, then $A^{-1} = (\det A)^{-1} \operatorname{adj} A$. Often the adjugate matrix is used to derive Cramer's rule for solving linear systems of equations by determinants. The proof given for the Cayley-Hamilton theorem, which states that every square matrix is a zero of its own characteristic polynomial, often makes use of the adjugate matrix, c.f. [**2**, p. 178].

In this paper we consider the adjugate in more detail and, in particular, we study adjugates of matrices that need not have inverses at all. We will note in particular that although adj A is $n \times n$ when A is $n \times n$, if $n \neq 2$, it is not true that every $n \times n$ matrix is an adjugate.

We make the following definitions and observations. If $n > 1$ and $A = (a_{ij})$ is an $n \times n$ complex matrix, the adjugate of A, $\operatorname{adj} A = (A_{ij})$, is the $n \times n$ complex matrix whose entries are $A_{ij} = (-1)^{i+j} \det A(j|i)$, $i, j = 1, \dots, n$, where for each $i, j = 1, \dots, n$, $A(j|i)$ is the $(n-1) \times (n-1)$ matrix that results from the deletion of the jth row and ith column of A. It is well known that $A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)I_n$, where I_n is the $n \times n$ identity matrix. In fact, this formula is the one required to derive the result $A^{-1} = (\det A)^{-1} \operatorname{adj} A$ when A^{-1} exists. To make the formula valid for $n = 1$, we put $\operatorname{adj} A = (1)$ for every 1×1 matrix A. For any positive integer n, we denote the set of $n \times n$ complex matrices by $M_n(\mathbb{C})$. We use $r(A)$ to denote the rank of A.

The following shows that adj is not *onto* $M_n(\mathbb{C})$ if $n \neq 2$.

THEOREM. *Let* $A \in M_n(\mathbb{C})$.
(a) $r(A) = n$ *implies that* $r(\operatorname{adj} A) = n$;
(b) $r(A) = n - 1$ *implies that* $r(\operatorname{adj} A) = 1$;
(c) $r(A) < n - 1$ *implies that* $r(\operatorname{adj} A) = 0$.

Proof. (a) This is clear from the formula $(\operatorname{adj} A)A = (\det A)I_n$. (b) If $r(A) = n - 1$ where $n > 1$, A has a nonsingular $(n-1) \times (n-1)$ submatrix. Thus $\operatorname{adj} A \neq 0$ and so $r(\operatorname{adj} A) \geq 1$. Furthermore $(\operatorname{adj} A)A = (\det A)I_n = 0$, and so there are $n - 1$ linearly independent columns of A in $\ker(\operatorname{adj} A)$. Thus $r(\operatorname{adj} A) = n - \dim \ker(\operatorname{adj} A) \leq n - (n - 1) = 1$, and so $r(\operatorname{adj} A) = 1$. In case $n = 1$, (b) holds by definition. (c) If $r(A) < n - 1$, $\operatorname{adj} A = 0$ and thus $r(\operatorname{adj} A) = 0$.

We note that for $n = 2$, adj is onto $M_n(\mathbb{C})$ since

$$\operatorname{adj} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Of course adj is not onto $M_n(\mathbb{C})$ when $n=1$ since $\operatorname{adj} A$ can only be the matrix (1).

We will now show that if $n>2$, although adj is not onto all of $M_n(\mathbb{C})$ due to the rank restrictions, adj is onto the set of matrices in $M_n(\mathbb{C})$ of rank n, 1, or 0. To establish this fact, and to make some observations concerning the form of the adjugate we shall make use of the following elementary properties, all of which are well known.

LEMMA. *For any* $A \in M_n(\mathbb{C})$,
(a) $\det(\operatorname{adj} A) = (\det A)^{n-1}$;
(b) $\operatorname{adj}(\operatorname{adj} A) = (\det A)^{n-2}A$;
(c) *for* P *nonsingular in* $M_n(\mathbb{C})$, $\operatorname{adj}(P^{-1}AP) = P^{-1}(\operatorname{adj} A)P$;
(d) *for* λ *complex*, $\operatorname{adj}(\lambda A) = \lambda^{n-1}\operatorname{adj} A$;
(e) $\operatorname{adj} A$ *is a polynomial in* A.

To give certain of these formulas and their proofs the correct meanings when $n=1$ or 2 we will find it necessary to put 0^0 and $\frac{0}{0}$ equal to 1. (For example, consider (d) for $n=1$ and $\lambda=0$).

Proof. (d) follows directly from the definition of adjugate. We prove the other parts for A nonsingular and then argue the general case by continuity: Every $A \in M_n(\mathbb{C})$ is the limit of a sequence of *nonsingular* matrices in $M_n(\mathbb{C})$. Thus we prove the identity for nonsingular A and then take limits. For example, in (a), suppose that $A_k \to A$ as $k \to \infty$, where each A_k is nonsingular. If (a) holds when A is nonsingular, $\det(\operatorname{adj} A_k) = (\det A_k)^{n-1}$ for each k. If we let $k \to \infty$, we get $\det(\operatorname{adj} A) = (\det A)^{n-1}$, even if A is singular. In the proofs that follow we assume $n>2$. The cases $n=1,2$ can be argued directly. We recall that when A is nonsingular, $\operatorname{adj} A = (\det A)A^{-1}$.
(a) $\det(\operatorname{adj} A) = \det[(\det A)A^{-1}] = (\det A)^n \cdot \det A^{-1} = (\det A)^{n-1}$;
(b) $\operatorname{adj}(\operatorname{adj} A) = [\det(\operatorname{adj} A)][\operatorname{adj} A]^{-1} = (\det A)^{n-1}[(\det A)A^{-1}]^{-1} = (\det A)^{n-2}A$;
(c) $\operatorname{adj}(P^{-1}AP) = \det(P^{-1}AP)\cdot(P^{-1}AP)^{-1} = (\det A)\cdot(P^{-1}A^{-1}P) = P^{-1}[(\det A)A^{-1}]P = P^{-1}(\operatorname{adj} A)P$.
(e) Suppose A has characteristic polynomial

$$f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

Then $a_0 = (-1)^n \det A$, and so, since $f(A)=0$,

$$A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n + (-1)^n(\det A)A^{-1} = 0,$$

i.e.,

$$\operatorname{adj} A = (-1)^{n-1}\left(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n\right).$$

In the same way as we argued in each of parts (a), (b), and (c), we conclude that since the entries in $\operatorname{adj} A$ and the coefficients in $f(\lambda)$ depend continuously on the entries of A, this last formula is also valid in case A is singular. Thus for any $A \in M_n(\mathbb{C})$, $\operatorname{adj} A = q(A)$, where $q(\lambda)$ is the polynomial

$$(-1)^{n-1}\lambda^{-1}\{f(\lambda) - f(0)\} = (-1)^{n-1}\left(\lambda^{n-1} + a_{n-1}\lambda^{n-2} + \cdots + a_1\right).$$

We now give some explicit constructions to show that adj is onto the members of $M_n(\mathbb{C})$ of ranks n, 1 or 0 when $n>2$. First suppose that $B \in M_n(\mathbb{C})$ and $r(B)=n$, where $n>2$. Put

$$A = (\det B)^{-(n-2)/(n-1)}\operatorname{adj} B.$$

Then

$$\operatorname{adj} A = (\det B)^{-(n-2)} \operatorname{adj}(\operatorname{adj} B) = (\det B)^{-(n-2)} (\det B)^{n-2} B = B.$$

Next suppose $B \in M_n(\mathbb{C})$ and $r(B) = 1$, $n > 2$. Consider first the case for which trace $B = \operatorname{tr} B \neq 0$. Since $\dim(\ker B) = n - 1$, B is similar to a diagonal matrix that may be taken as $\operatorname{diag}(\operatorname{tr} B, 0, \dots, 0)$.

If $X \in M_m(\mathbb{C})$ and $Y \in M_{n-m}(\mathbb{C})$, where $m < n$, one writes

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} = X \oplus Y,$$

the *direct sum* of X and Y. The zeros denote $m \times (n-m)$ and $(n-m) \times m$ zero matrices in the first and second row, respectively. Thus, one could write, for example, $P^{-1}BP = (\operatorname{tr} B) \oplus 0_{n-1}$, where 0_{n-1} is the $(n-1) \times (n-1)$ zero matrix. Put

$$A = (\operatorname{tr} B)^{-(n-2)/(n-1)} \left[(\operatorname{tr} B) I_n - B \right].$$

Then

$$\begin{aligned} P^{-1}AP &= (\operatorname{tr} B)^{-(n-2)/(n-1)} \{ (\operatorname{tr} B) I_n - [(\operatorname{tr} B) \oplus 0_{n-1}] \} \\ &= (\operatorname{tr} B)^{-(n-2)/(n-1)} \left[0 \oplus (\operatorname{tr} B) I_{n-1} \right], \end{aligned}$$

and so

$$\begin{aligned} P^{-1}(\operatorname{adj} A) P = \operatorname{adj}(P^{-1}AP) &= (\operatorname{tr} B)^{-(n-2)} \operatorname{adj}\left[0 \oplus (\operatorname{tr} B) I_{n-1} \right] \\ &= (\operatorname{tr} B)^{-(n-2)} \left[(\operatorname{tr} B)^{n-1} \oplus 0_{n-1} \right] = (\operatorname{tr} B) \oplus 0_{n-1} = P^{-1}BP. \end{aligned}$$

Thus $\operatorname{adj} A = B$. We now consider the case for which $B \in M_n(\mathbb{C})$ and $r(B) = 1$, $n > 2$, but for which $\operatorname{tr} B = 0$. Since we still have $\dim(\ker B) = n - 1$, B must have a Jordan form

$$B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus 0_{n-2}.$$

We write $B_1 = P^{-1}BP$. For

$$A_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \oplus I_{n-2}$$

put $A = PA_1P^{-1}$. Then $\operatorname{adj} A = P(\operatorname{adj} A_1)P^{-1} = PB_1P^{-1} = B$.

Lastly, consider the case $r(B) = 0$. Then $B = 0$ and if $A = 0$ and $n > 2$, of course $B = \operatorname{adj} A$. In fact, for any A of rank less than $n - 1$, $B = \operatorname{adj} A = 0$.

We note that in each case, except possibly that in which $r(B) = 1$ and $\operatorname{tr} B = 0$, A may be chosen so that it is an elementary function, and in fact, a polynomial, in B. Consider once more the case $r(B) = 1$, $\operatorname{tr} B = 0$, where $n > 2$. For $Q_1 = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right] \oplus (-I_{n-2})$, $A_1 = \frac{1}{2} I_n - \frac{1}{2} Q_1$, and

$$Q_1^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \oplus I_{n-2} = I_n + 4B_1.$$

Thus $Q^2 = I_n + 4B$, where $Q = PQ_1P^{-1}$. The matrix Q is a *square root* of $I_n + 4B$. If we use the notation $Q = \sqrt{I_n + 4B}$ we can write

$$A = \frac{1}{2} I_n - \frac{1}{2} Q = \frac{1}{2} I_n - \frac{1}{2} \sqrt{I_n + 4B}$$

for this particular square root Q of $I_n + 4B$.

In spite of the elegant form of this A, one might ask if this matrix A, or any solution A, is a polynomial in B as was the situation in each of the other cases considered for B. The matrix A would be a polynomial in B if the matrix Q above were a polynomial in B; Q would be such a polynomial if $-I_{n-2}$ were replaced by $+I_{n-2}$ in Q_1, for example. However, such a Q_1 and resulting $Q = I_n + 2B$, another square root of $I_n + 4B$, could never be used to determine an appropriate matrix A. We shall argue that no A exists that is a polynomial in B such that $B = \operatorname{adj} A$. Thus no square root of $I_n + 4B$ that is a polynomial in B could be used.

Since $B^k = PB_1^kP^{-1} = 0$ for any $k > 1$, if $B = \operatorname{adj} A$ for $A = p(B)$, a polynomial in B, $p(\lambda)$ must be of the form $a + b\lambda$. Then

$$A = p(B) = p\left(PB_1P^{-1}\right) = Pp(B_1)P^{-1} = P\left[\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \oplus aI_{n-2}\right]P^{-1}.$$

Thus

$$\begin{aligned} B = \operatorname{adj} A &= P \cdot \operatorname{adj}\left[\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \oplus aI_{n-2}\right] \cdot P^{-1} \\ &= P\left[\begin{bmatrix} a^{n-1} & -ba^{n-2} \\ 0 & a^{n-1} \end{bmatrix} \oplus a^{n-1}I_{n-2}\right]P^{-1}. \end{aligned}$$

Since $0 = \operatorname{tr} B = na^{n-1}$, $a = 0$ and so $B = 0$. But then $r(B)$ cannot be 1. It follows that if $r(B) = 1$ and $\operatorname{tr} B = 0$, B is the adjugate of *no* polynomial in B.

Example.

$$B = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

has $r(B) = 1$, $\operatorname{tr} B = 0$ with $n = 3$.

We can reduce B to Jordan form

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ -2 & 1 & 0 \end{bmatrix}; \qquad B_1 = P^{-1}BP.$$

Then with

$$Q_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \sqrt{I_3 + 4B} = PQ_1P^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ 8 & -5 & -4 \\ -6 & 4 & 3 \end{bmatrix}$$

and so

$$A = \frac{1}{2}I_3 - \frac{1}{2}\sqrt{I_3 + 4B} = \begin{bmatrix} -1 & 1 & 1 \\ -4 & 3 & 2 \\ 3 & -2 & -1 \end{bmatrix}.$$

One may check that $B = \operatorname{adj} A$.

A is not of the form $aI_3 + bB$ and so A is not a polynomial in B. However, B *is* a polynomial in A: The above formula for A leads to $B = A^2 - A$. The same result is obtained from the formula in the proof of part (e) of the Lemma.

We conclude the paper with comments concerning the cases $n = 1$ and $n = 2$.

If $n = 1$ and $B = (1)$, then for any $A = (a)$, $B = \operatorname{adj} A$ and $A = aB$, a polynomial in B. If $B \neq (1)$, $B = \operatorname{adj} A$ for no matrix A.

If $n = 2$, $B = \operatorname{adj} A$ if and only if $A = \operatorname{adj} B = (\operatorname{tr} B)I_2 - B$, whatever be the values of $r(B)$ and $\operatorname{tr} B$.

Note. This paper is dedicated to my beloved grandson, James P. Busbee, Jr.

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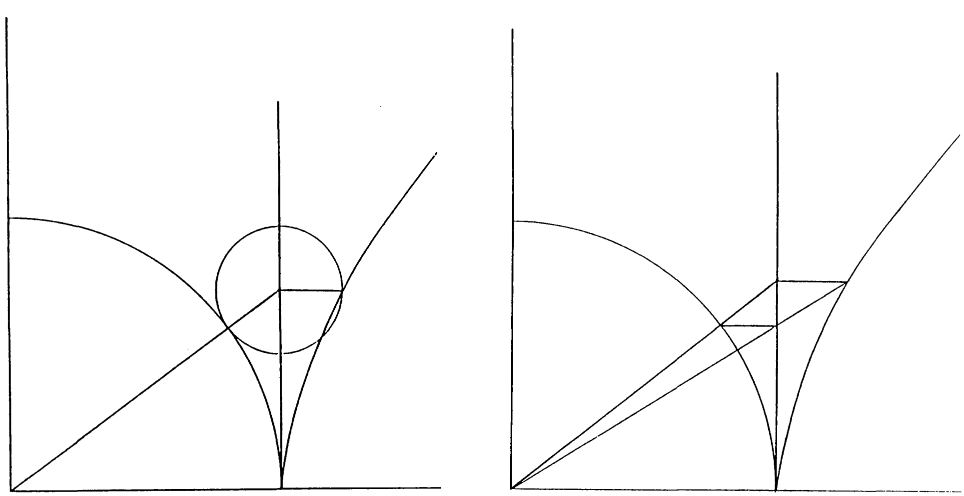
Constructions without Words



Construction of a Hyperbola I   Construction of a Hyperbola II

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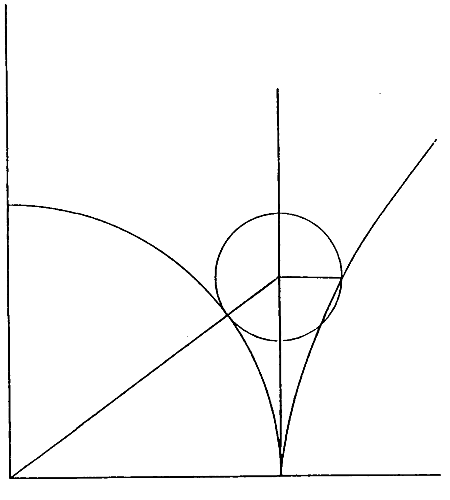
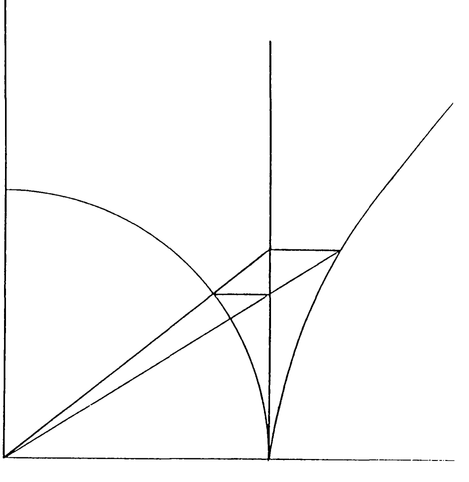


Construction of a Hyperbola I



Construction of a Hyperbola II

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Subgroups and the Partitioning Property

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Lagrange's theorem states that the order of a subgroup H of a finite group G divides the order of G. The key step in the standard proof is to show that the left cosets *partition* the group, that is, any two left cosets of the subgroup are either equal or disjoint and the union of all the left cosets is the whole group. In this note, we answer the question: *Are there subsets other than subgroups whose "left cosets" partition G?*

Let X be a subset of elements of a group G. If g is an element of G, the *left translate* gX of X is the set $\{gx: x \in X\}$. A collection $\mathscr{C}$ of nonempty subsets of G is said to *partition* G if any two subsets in $\mathscr{C}$ are either equal or disjoint and their union is G.

THEOREM. *Let X be a subset of a group G. Then the left translates of X partition G if and only if X is a (left or right) coset of a subgroup of G.*

Proof. Suppose first that X is a coset of a subgroup H of G. If X is a right coset and equals Ha, where $a \in G$, then $X = Ha = a(a^{-1}Ha)$. Hence, X is a left coset of the conjugate subgroup $a^{-1}Ha$. Thus, we can suppose that X is a left coset of H and X equals aH. Since $gX = gaH$ and $gH = ga^{-1}aH = ga^{-1}X$, left translates of X are left cosets of H and conversely. We conclude that the left translates of X partition G.

Now suppose that the left translates of X partition G. Let H be the left translate aX containing the identity e of G. Because $gX = ga^{-1}aX = ga^{-1}H$, left translates of H coincide with left translates of X. Therefore, X itself is a left translate of H, the left translates of H partition G, and it remains to show that H is a subgroup of G. Let x and y be elements in H. Because y is in H, the product xy is in xH. Moreover, as e is in H and $x = xe, x$ is in xH. Hence, H and xH are not disjoint and xH equals H. We conclude that xy is in H. Now consider $x^{-1}x$. Since it equals the identity e, it is in H. On the other hand, since x is in $H, x^{-1}x$ is in $x^{-1}H$. Hence, H and $x^{-1}H$ are not disjoint and are equal. We conclude that x^{-1} is in H.

One way in which a partition arises naturally is as the collection of nonempty inverse images $f^{-1}(t)$ of a function f: $E \to S$. In particular, let G be a group of permutations acting on the set S and let s be a fixed element of S. Consider the function f: $G \to S$ defined by $f(\pi) = \pi(s)$, for $\pi \in G$. An inverse image $f^{-1}(t)$ is nonempty if and only if t is in the orbit of s, that is, t equals $\rho(s)$ for some permutation ρ in G. For such an element t and a permutation σ in G,

$$\sigma[f^{-1}(t)] = \{\sigma\pi: \pi(s) = t\} = \{\tau: \tau(s) = \sigma(t)\} = f^{-1}(\sigma(t))$$

and

$$f^{-1}(\sigma(s)) = f^{-1}(\sigma\rho^{-1}\rho(s)) = \sigma\rho^{-1}[f^{-1}(t)].$$

Thus, the nonempty inverse images coincide with the left translates of $f^{-1}(t)$. We conclude that the left translates of $f^{-1}(t)$ partition G. By the theorem, they are the cosets of a subgroup. This subgroup is the left translate $f^{-1}(s)$ consisting of the permutations π such that $\pi(s) = s$. We see, then, that the subgroup is the stabilizer $\mathrm{Stab}(s)$ of s. It follows that the number of elements in the orbit of $s = |G|/|\mathrm{Stab}(s)|$.

A similar argument can be used to show that the kernel of a homomorphism is a subgroup.

When a group G can be depicted geometrically, it is often easy to see whether left translates partition G. Consider the group $\mathbb{C}^\times$ of nonzero complex numbers under multiplication. If $X \subset \mathbb{C}^\times$, then the left translate $re^{i\theta}X$ can be obtained geometrically from X by rotating through the angle θ and dilating (or "stretching") by r. Suppose X is a closed curve (that is, a continuous image of the circle containing at least two points). Then it is intuitively evident that X and a rotation of X through a sufficiently small angle intersect except in the case when X is a circle. (See FIGURE 1.) Using the fact that X is compact, this intuition can be formalized. Thus, except for circles (centered at 0), closed curves are not cosets of subgroups. On the other hand, the left translates of a circle partition $\mathbb{C}^\times$ and are the cosets of the subgroup consisting of the unit circle. Using this fact, we obtain the following result.

THEOREM. *Let H be a subgroup of $\mathbb{C}^\times$ containing a closed curve. Then H is either the whole group $\mathbb{C}^\times$ or the unit circle.*

The condition that the curve be closed is crucial. For example, the unions of open half-lines shown in FIGURE 2 are subgroups since their left translates partition $\mathbb{C}^\times$. However, cosets can partition $\mathbb{C}^\times$ in a nonobvious way. An example is given by the "spiral" subgroup $\{re^{i \log r}: r \in \mathbb{R}\}$.

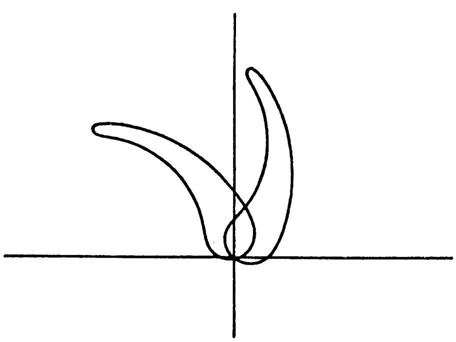

FIGURE 1
A closed curve and its rotation.

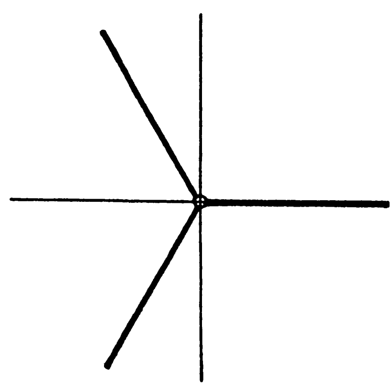

FIGURE 2
A subgroup consisting of three half-lines. The angle between any two lines is 120°.

We end this note with an observation. The theorem holds if we use *right* translates rather than left translates. Thus, it follows that the right translates of a set X partition G if and only if the left translates partition G. It seems relatively difficult (and not elementary) to prove this fact directly without first proving that X is a coset of a subgroup.

A Problem of Selection of Points from the Unit Interval

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1. Introduction There are many problems in probability for which the solutions seem to defy intuition. The following problem involving a selection of points from the interval $(0,1)$ is one of these. Although this problem can be attacked with more sophisticated probabilistic machinery (see Section 4), it can also be solved with some elementary knowledge of conditional probability and a little calculus.

Imagine the following scheme for selecting points from the interval $(0,1)$. Initially a person is asked to select a point, x_1, at random from the interval, after which a second person selects a point at random from the interval $(0, x_1)$. The process continues with the nth person choosing a point at random from the interval $(0, x_{n-1})$, and so on. Suppose that you happen to arrive after the selection process has been going on for some time, and are only allowed to see the value of the point, x_n, that has been selected just prior to your arrival. With this information what can you deduce about the value of the first point selected?

If x_n is a value close to 1 then you can conclude that x_1 must be a value even closer to 1. The intuition problems arise when the value of x_n is close to 0. For example, if n is 100, then it is possible that one of the values $x_2, x_3, \ldots, x_{99}$ was a small value and forced x_{100} to be small, while x_1 could still have been a reasonably large value. Should this small value of x_{100} have any influence on your guess at the value of x_1?

Intuitively, if n is large it seems quite likely that at least one of the points $x_2, x_3, \ldots, x_{n-1}$ will be a value "close" to 0. Therefore, it can be argued that a small value of x_n should not make x_1 the culprit. Perhaps it is, but why not blame the smallness of x_n on a previously drawn small value of $x_2, x_3, \ldots, x_{n-1}$, since it seems so likely that one of these values (and hence all those selected afterwards) is small?

The solution to the above problem by elementary methods is the focus of this article. In Section 2 an answer is obtained through analysis of some functions that are developed in Section 3. Section 4 contains some remarks and possible alternative ways of viewing the selection process.

2. An answer to the question The selection process described in the introduction can be modeled in the following manner. Let $X_1, X_2, X_3, \ldots$ be random variables such that X_1 is uniformly distributed on the interval $(0,1)$, and for $n \geq 2$, X_n is conditionally uniformly distributed on the interval $(0, x_{n-1})$ given that $X_{n-1} = x_{n-1}$. The problem in question centers around $E(X_1|X_n = x_n)$, the conditional expectation of X_1, given that $X_n = x_n$. Although it is possible to answer the question by considering other objects, obtaining an expression for $E(X_1|X_n = x_n)$ seems to be the most direct and illuminating approach. For notational convenience the subscript on x_n will be dropped and the function $g_n(x) = E(X_1|X_n = x)$ defined. In the following section it will be shown that for $0 < x < 1$ and $n \geq 2$,

$$g_n(x) = E(X_1|X_n = x) = \frac{(n-1)!}{[\ln(x)]^{n-1}}\left\{x - \sum_{k=0}^{n-2} (\ln x)^k/k!\right\}. \tag{1}$$

For now, assume that (1) holds and consider some properties of the functions $g_n(x)$. A graph of $g_n(x)$ for $n = 2, 3, 4, 5$, and 6 is given in FIGURE 1 and some values of the functions $g_n(x)$ for various choices of x in $(0, 1)$ appear in TABLE 1.

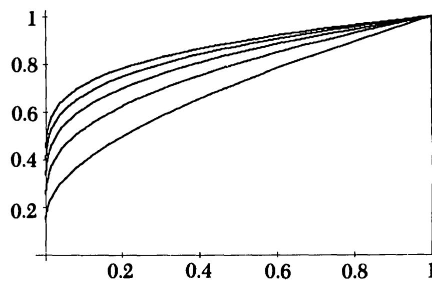

FIGURE 1

Graphs of $g_n(x)$ for $n = 2, 3, \ldots, 6$. In the picture, $g_2(x) \leq g_3(x) \leq \cdots \leq g_6(x)$.

TABLE 1. VALUES OF $g_n(x)$ FOR VARIOUS VALUES OF n AND x.

	$x = 0.05$	$x = 0.1$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$
$n = 2$	0.317	0.391	0.497	0.655	0.783	0.896
$n = 3$	0.456	0.529	0.625	0.753	0.849	0.930
$n = 4$	0.545	0.614	0.699	0.807	0.884	0.947
$n = 5$	0.608	0.671	0.748	0.841	0.906	0.957
$n = 6$	0.655	0.714	0.783	0.865	0.921	0.964
$n = 7$	0.691	0.746	0.809	0.883	0.931	0.969
$n = 8$	0.721	0.772	0.830	0.896	0.940	0.973
$n = 9$	0.745	0.793	0.846	0.907	0.946	0.976
$n = 10$	0.766	0.810	0.860	0.915	0.951	0.978

The graphs of $g_1(x)$ through $g_6(x)$ in FIGURE 1 suggest that the family of functions $g_n(x)$ for $n = 1, 2, 3, \ldots$ satisfies the three properties given below. The first property states that for $n \geq 2$ fixed, the expected value of X_1 given that $X_n = x$ goes to 0 as x approaches 0 from the right, thus answering the question posed in the introduction. Note however from FIGURE 1 and TABLE 1, that even for relatively small values of n, $E(X_1 | X_n = x)$ remains greater than $\frac{1}{2}$ until x is extremely small.

Property 1. For $n \geq 2$ fixed, $\lim_{x \to 0+} E(X_1 | X_n = x) = 0$.

Proof. Again, assuming that $g_n(x)$ is defined by (1), it follows that

$$\lim_{x \to 0+} g_n(x) = \lim_{x \to 0+} \frac{x - \left(1 + \dfrac{\ln x}{1!} + \cdots + \dfrac{(\ln x)^{n-2}}{(n-2)!}\right)}{\dfrac{(\ln x)^{n-1}}{(n-1)!}}$$

$$= (n-1)! \lim_{x \to 0+} \left\{ \frac{(x-1)}{(\ln x)^{n-1}} - \frac{1}{1!(\ln x)^{n-2}} - \cdots - \frac{1}{(n-2)! \ln x} \right\}$$

$$= 0,$$

since each of the terms goes to 0 for n fixed.

The second property records the observation that for each $n \geq 2$, as x approaches 1 from the left, the expected value of X_1, given $X_n = x$, approaches 1.

Property 2. For $n \geq 2$ fixed, $\lim_{x \to 1-} E(X_1 | X_n = x) = 1$.

Proof. This follows immediately from the fact that $X_n = x < X_1 < 1$, and can also be obtained analytically from (1).

The third property states that for $0 < x < 1$ fixed, the expected value of X_1 given $X_n = x$ approaches 1 as n goes to infinity.

Property 3. For $0 < x < 1$ fixed, $\lim_{n \to \infty} E(X_1 | X_n = x) = 1$.

Proof. Using the fact that $x = \Sigma_{k=0}^{+\infty} (\ln x)^k / k!$ and $g_n(x)$ is defined by (1), it follows that

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x - \sum_{k=0}^{n-2} (\ln x)^k / k!}{\dfrac{(\ln x)^{n-1}}{(n-1)!}}$$

$$= \lim_{n\to\infty} \frac{\sum_{k=n-1}^{+\infty} (\ln x)^k / k!}{\dfrac{(\ln x)^{n-1}}{(n-1)!}}$$

$$= \lim_{n\to\infty} \left\{ 1 + \sum_{k=1}^{+\infty} \frac{(\ln x)^k}{n(n+1)\cdots(n+k-1)} \right\}. \tag{2}$$

Now suppose for a moment that n is also fixed and large enough so that $n + 1 \geqslant |\ln x|$. For such an n, the series

$$\sum_{k=1}^{+\infty} \frac{(\ln x)^k}{n(n+1)\cdots(n+k-1)}$$

is an alternating series whose terms decrease in magnitude. Therefore it follows that

$$-\left|\frac{\ln x}{n}\right| \leq \sum_{k=1}^{+\infty} \frac{(\ln x)^k}{n(n+1)\cdots(n+k-1)} \leq \left|\frac{\ln x}{n}\right|. \tag{3}$$

Now letting n go to infinity in (3) yields

$$\lim_{n\to\infty} \sum_{k=1}^{+\infty} \frac{(\ln x)^k}{n(n+1)\cdots(n+k-1)} = 0,$$

and substituting this into (2) gives $\lim_{n \to \infty} g_n(x) = 1$.

3. A method of obtaining $g_n(x)$ using elementary probability and calculus It will now be shown how the expression (1) can be obtained using some rudimentary ideas from conditional probability and calculus.

Let $X_1, X_2, \ldots$ be random variables such that X_1 is uniformly distributed on the interval $(0, 1)$ and for $n \geq 2$, X_n is conditionally uniform on $(0, x_{n-1})$ given that $X_{n-1} = x_{n-1}$. The density of X_1 is given by $f(x_1) = 1$ for $0 < x_1 < 1$ and 0 otherwise; the conditional density of X_2 given X_1 is $f(x_2 | x_1) = 1/x_1$ for $0 < x_2 < x_1 < 1$ and 0 otherwise. The conditional density of X_n given $X_1, \ldots, X_{n-1}$ is given by $f(x_n|x_1, x_2, \ldots, x_{n-1}) = f(x_n|x_{n-1}) = 1/x_{n-1}$ for $0 < x_n < x_{n-1} < \cdots < x_1 < 1$ and 0 otherwise. Using this information it is possible to obtain the joint distribution of x_1 through x_n for $n \geq 2$,

$$f(x_1, \ldots, x_n) = f(x_n|x_{n-1}) f(x_{n-1}|x_{n-2}) \ldots f(x_2|x_1) f(x_1)$$
$$= \frac{1}{x_1 x_2 \ldots x_{n-1}} \quad \text{for } 0 < x_n < x_{n-1} < \cdots < x_1 < 1, \quad \text{and 0 otherwise.}$$

The conditional distribution of X_1 given that $X_n = x_n$ is given by the formula

$$f(x_1|x_n) = \frac{f(x_1, x_n)}{f(x_n)}. \tag{4}$$

The numerator of (4) can be calculated for $n \geq 3$ as follows:

$$f(x_1, x_n) = \int_{x_n}^{x_1} \int_{x_n}^{x_2} \cdots \int_{x_n}^{x_{n-2}} \frac{1}{x_1 x_2 \ldots x_{n-1}} \, dx_{n-1} \, dx_{n-2} \ldots dx_2$$
$$= \frac{1}{x_1 (n-2)!} \left[\ln\left(\frac{x_1}{x_n}\right) \right]^{n-2} \quad \text{for } 0 < x_n < x_1 < 1, \tag{5}$$

although the formula (5) is also valid when $n = 2$. The denominator of (4) can be calculated merely by integrating (5) with respect to x_1,

$$f(x_n) = \int_{x_n}^{1} \frac{1}{x_1 (n-2)!} \left[\ln\left(\frac{x_1}{x_n}\right) \right]^{n-2} dx_1$$
$$= \frac{1}{(n-1)!} \left[\ln\left(\frac{1}{x_n}\right) \right]^{n-1}, \quad \text{for } 0 < x_n < 1, \tag{6}$$

valid for all $n \geq 2$. Now substituting (5) and (6) into (4) yields for $n \geq 2$,

$$f(x_1|x_n) = \frac{(n-1)[\ln(x_1/x_n)]^{n-2}}{x_1 [\ln(1/x_n)]^{n-1}} \quad \text{for } 0 < x_n < x_1 < 1, \quad \text{0 otherwise.} \tag{7}$$

Using (7) and the definition of $E(X_1|X_n = x_n)$ it follows that for $0 < x_n < 1$, and $n \geq 2$,

$$E(X_1|X_n = x_n) = \int_{-\infty}^{+\infty} x_1 f(x_1|x_n) \, dx_1$$
$$= \frac{(n-1)}{[\ln(1/x_n)]^{n-1}} \int_{x_n}^{1} [\ln(x_1/x_n)]^{n-2} \, dx_1. \tag{8}$$

The integral in (8) can be solved using repeated integration by parts to obtain for $n \geq 3$,

$$E(X_1|X_n = x_n) = \frac{(n-1)}{\ln(1/x_n)} - \frac{(n-1)(n-2)}{[\ln(1/x_n)]^2} + \frac{(n-1)(n-2)(n-3)}{[\ln(1/x_n)]^3} - \cdots$$
$$+ (-1)^{n-1}(n-1)! \left\{ \frac{1}{[\ln(1/x_n)]^{n-2}} - \frac{(1-x_n)}{[\ln(1/x_n)]^{n-1}} \right\}. \tag{9}$$

Suppressing the subscript on x_n, and using simple algebraic manipulation (9) can be expressed more succinctly as

$$g_n(x) = E(X_1|X_n = x) = \frac{(n-1)!}{[\ln(x)]^{n-1}}\left\{x - \sum_{i=0}^{n-2}\frac{(\ln(x))^i}{i!}\right\},$$

which is expression (1) from Section 2.

4. Alternate ways of viewing the selection process Readers familiar with the Poisson process may appreciate the following, more elegant approach.

Let $X_1, X_2, \ldots$ be as given in Section 3, and define for each $n \geq 1$ the sequence of random variables $Y_1, Y_2, \ldots$ by $Y_n = -\ln X_n$. It can be easily shown that $Y_1, Y_2 - Y_1, Y_3 - Y_2, \ldots$ is a sequence of independent and identically-distributed, exponential-parameter 1 random variables. Note then that since $Y_n = Y_1 + (Y_2 - Y_1) + \cdots + (Y_n - Y_{n-1})$, it follows that Y_n has a gamma distribution with parameters $\alpha = n$ and $\beta = 1$. The random variables $Y_1, Y_2, \ldots$ can be used to model the arrival times of a Poisson process and given the nth arrival time $Y_n = y_n$ it can be shown that $Y_1, Y_2, \ldots, Y_{n-1}$ are distributed like the order statistics of a sample of size $(n-1)$ from a uniform distribution on the interval $(0, y_n)$. Thus the conditional density of Y_1 given Y_n is

$$f(y_1|y_n) = (n-1)\left(1 - \frac{y_1}{y_n}\right)^{n-2}\left(\frac{1}{y_n}\right) \quad \text{for } 0 < y_1 < y_n, \tag{10}$$

which can be used to obtain (7). Note from (10) it follows that $E(Y_1|Y_n = y_n) = (y_n/n)$. Property 1 can be obtained directly by showing that for each $n \geq 2$, the conditional distribution of Y_1, given $Y_n = y_n$, converges in distribution to $+\infty$ as y_n goes to $+\infty$, thus the conditional distribution of X_1, given $X_n = x_n$, converges in distribution to 0 as x_n goes to 0. Since X_1 is bounded its conditional mean, $E(X_1|X_n = x_n)$ must also go to 0. The question posed in the introduction now becomes "given the time of the nth arrival of a Poisson process, what should you expect the time of the first arrival to have been?" [This approach was shown to the author both by a colleague, R. M. Norton, and an anonymous referee. Although more sophisticated, it does not necessarily relieve the intuition difficulties.]

It is also worthwhile noting that the selection process described in the introduction can be viewed in yet another fashion. Let $X_1, X_2, \ldots$ be independent identically distributed random variables having a uniform distribution on the interval (0, 1). For each $j \geq 1$, set $Y_j = \prod_{i=1}^{j} X_i$. It can be shown that the distribution of the sequence $Y_1, Y_2, \ldots$ is the same as that given for the X_j's as defined in the introduction. The question there posed now becomes "given the value of the product of n randomly chosen numbers from the interval (0, 1), what is the expected value of the first number selected?" This formulation differs only slightly from the original, but may be useful in the construction of a simulation of the process.

Acknowledgements. The author first became aware of this problem while teaching a course in Mathematical Statistics. The $n = 2$ case appears as part of a problem in *Introduction to Mathematical Statistics*, 4th edition, by Hogg and Craig (problem 2.16). The author would like to thank Dr. Robert M. Norton, Dr. Reginald Koo, and Dr. Ted Hill, and the anonymous referees who contributed a great deal to improving the quality of this article.

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A Menelaus-Type Theorem for the Pentagram

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A pentagram, or star-polygon, is formed by the diagonals of a convex pentagon such as that shown in FIGURE 1. The *regular* pentagram has been studied extensively beginning with the Pythagoreans who used it as an emblem of their society. The most ubiquitous property of the regular pentagram is that involving the golden ratio [**1**]. A very brief history of regular pentagrams is given in [**3**, pp. 44–45]. In this note we derive some additional properties for arbitrary pentagrams.

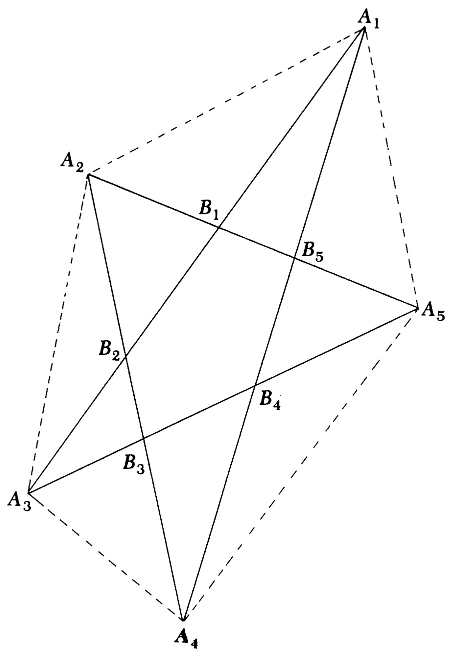

FIGURE 1

Our main result is reminiscent of the theorems of Menelaus and Ceva in that the products of ratios of segments taken in order around a polygon is equal to ± 1. More precisely:

THEOREM 1. *If* $A_1B_1A_2B_2A_3B_3A_4B_4A_5B_5$ *is a pentagram, then*

$$\frac{A_1B_1}{B_1A_2} \cdot \frac{A_2B_2}{B_2A_3} \cdot \frac{A_3B_3}{B_3A_4} \cdot \frac{A_4B_4}{B_4A_5} \cdot \frac{A_5B_5}{B_5A_1} = 1.$$

For the proof we repeatedly use Menelaus' Theorem, which states that if a line intersects the (extended) sides of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ of ΔABC in points D, E, F, respectively, then

$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = -1.$$

Proofs of this famous theorem can be found in many geometry books—one example is [2].

For our proof we find it convenient to let $A_{i+5}=A_i$ and $B_{i+5}=B_i$ for $i=1,2,\ldots,5$. We use Menelaus' Theorem in blocks of five triangles as follows:

For triangles of the form $A_iB_iB_{i+4}$ and lines $\overline{A_{i+1}A_{i+3}}$, we have

$$\begin{aligned}
&\frac{A_1B_2}{B_2B_1}\cdot\frac{B_1A_2}{A_2B_5}\cdot\frac{B_5A_4}{A_4A_1}=-1,\\
&\frac{A_2B_3}{B_3B_2}\cdot\frac{B_2A_3}{A_3B_1}\cdot\frac{B_1A_5}{A_5A_2}=-1,\\
&\frac{A_3B_4}{B_4B_3}\cdot\frac{B_3A_4}{A_4B_2}\cdot\frac{B_2A_1}{A_1A_3}=-1,\\
&\frac{A_4B_5}{B_5B_4}\cdot\frac{B_4A_5}{A_5B_3}\cdot\frac{B_3A_2}{A_2A_4}=-1 \text{ and}\\
&\frac{A_5B_1}{B_1B_5}\cdot\frac{B_5A_1}{A_1B_4}\cdot\frac{B_4A_3}{A_3A_5}=-1.
\end{aligned} \tag{1}$$

For the same triangles $A_iB_iB_{i+4}$ but for lines $\overline{A_{i+2}A_{i+4}}$, we have

$$\begin{aligned}
&\frac{A_1A_3}{A_3B_1}\cdot\frac{B_1A_5}{A_5B_5}\cdot\frac{B_5B_4}{B_4A_1}=-1,\\
&\frac{A_2A_4}{A_4B_2}\cdot\frac{B_2A_1}{A_1B_1}\cdot\frac{B_1B_5}{B_5A_2}=-1,\\
&\frac{A_3A_5}{A_5B_3}\cdot\frac{B_3A_2}{A_2B_2}\cdot\frac{B_2B_1}{B_1A_3}=-1,\\
&\frac{A_4A_1}{A_1B_4}\cdot\frac{B_4A_3}{A_3B_3}\cdot\frac{B_3B_2}{B_2A_4}=-1 \text{ and}\\
&\frac{A_5A_2}{A_2B_5}\cdot\frac{B_5A_4}{A_4B_4}\cdot\frac{B_4B_3}{B_3A_5}=-1.
\end{aligned} \tag{2}$$

By multiplying together the ten equations of (1) and (2), simplifying, and rearranging factors we have

$$\left[\frac{A_1B_2}{B_2A_4}\cdot\frac{A_4B_5}{B_5A_2}\cdot\frac{A_2B_3}{B_3A_5}\cdot\frac{A_5B_1}{B_1A_3}\cdot\frac{A_3B_4}{B_4A_1}\right]^3\times\left[\frac{B_5A_1}{A_1B_1}\cdot\frac{B_1A_2}{A_2B_2}\cdot\frac{B_2A_3}{A_3B_3}\cdot\frac{B_3A_4}{A_4B_4}\cdot\frac{B_4A_5}{A_5B_5}\right]=1. \tag{3}$$

Similarly for triangles of the form $A_iA_{i+2}B_{i+3}$ and lines $\overline{A_{i+1}A_{i+4}}$, we have

$$\begin{aligned}
&\frac{A_1B_1}{B_1A_3}\cdot\frac{A_3A_5}{A_5B_4}\cdot\frac{B_4B_5}{B_5A_1}=-1,\\
&\frac{A_2B_2}{B_2A_4}\cdot\frac{A_4A_1}{A_1B_5}\cdot\frac{B_5B_1}{B_1A_2}=-1,\\
&\frac{A_3B_3}{B_3A_5}\cdot\frac{A_5A_2}{A_2B_1}\cdot\frac{B_1B_2}{B_2A_3}=-1,
\end{aligned} \tag{4}$$

$$\frac{A_4B_4}{B_4A_1}\cdot\frac{A_1A_3}{A_3B_2}\cdot\frac{B_2B_3}{B_3A_4}=-1 \text{ and}$$

$$\frac{A_5B_5}{B_5A_2}\cdot\frac{A_2A_4}{A_4B_3}\cdot\frac{B_3B_4}{B_4A_5}=-1.$$

For the same triangles $A_iA_{i+2}B_{i+3}$ but for lines $\overline{A_{i+1}A_{i+3}}$, we have

$$\frac{A_1B_2}{B_2A_3}\cdot\frac{A_3B_3}{B_3B_4}\cdot\frac{B_4A_4}{A_4A_1}=-1,$$

$$\frac{A_2B_3}{B_3A_4}\cdot\frac{A_4B_4}{B_4B_5}\cdot\frac{B_5A_5}{A_5A_2}=-1,$$

$$\frac{A_3B_4}{B_4A_5}\cdot\frac{A_5B_5}{B_5B_1}\cdot\frac{B_1A_1}{A_1A_3}=-1, \tag{5}$$

$$\frac{A_4B_5}{B_5A_1}\cdot\frac{A_1B_1}{B_1B_2}\cdot\frac{B_2A_2}{A_2A_4}=-1 \text{ and}$$

$$\frac{A_5B_1}{B_1A_2}\cdot\frac{A_2B_2}{B_2B_3}\cdot\frac{B_3A_3}{A_3A_5}=-1.$$

By multiplying the ten equations of (4) and (5), simplifying, and rearranging the factors, we obtain

$$\left[\frac{A_1B_2}{B_2A_4}\cdot\frac{A_4B_5}{B_5A_2}\cdot\frac{A_2B_3}{B_3A_5}\cdot\frac{A_5B_1}{B_1A_3}\cdot\frac{A_3B_4}{B_4A_1}\right]^3\times\left[\frac{A_1B_1}{B_1A_2}\cdot\frac{A_2B_2}{B_2A_3}\cdot\frac{A_3B_3}{B_3A_4}\cdot\frac{A_4B_4}{B_4A_5}\cdot\frac{A_5B_5}{B_5A_1}\right]=1. \tag{6}$$

By dividing equation (3) by (6), observing that the cubed factors are identical, and the second factors are reciprocals, we obtain

$$\left[\frac{A_1B_1}{B_1A_2}\cdot\frac{A_2B_2}{B_2A_3}\cdot\frac{A_3B_3}{B_3A_4}\cdot\frac{A_4B_4}{B_4A_5}\cdot\frac{A_5B_5}{B_5A_1}\right]^2=1$$

and the result follows. As a bonus of (3) and (6) we note that

$$\frac{A_1B_2}{B_2A_4}\cdot\frac{A_4B_5}{B_5A_2}\cdot\frac{A_2B_3}{B_3A_5}\cdot\frac{A_5B_1}{B_1A_3}\cdot\frac{A_3B_4}{B_4A_1}=1.$$

REFERENCES

1. H. V. Baravalle, The geometry of the pentagon and the golden section, *The Mathematics Teacher* 41 (1948), 22–31.
2. H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, MAA (New Mathematical Library), 1967.
3. Howard Eves, *A Survey of Geometry* (rev. ed.), Allyn and Bacon, Inc., Boston, 1972.

Exclusive Disjunction and the Biconditional: An Even-Odd Relationship

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An elementary truth table argument shows that exclusive disjunction is just the negation of the biconditional: $(P \oplus Q) \equiv \neg(P \Leftrightarrow Q)$. This relationship is sometimes used to explain why inclusive, rather than exclusive, disjunction is the standard disjunction. Either disjunction can be formed from the other $((P \vee Q) \equiv ((P \oplus Q) \oplus (P \wedge Q))$; $(P \oplus Q) \equiv ((P \vee Q) \wedge \neg(P \wedge Q)))$, but only exclusive disjunction is the negation of another simple connective.

However, while $P \oplus Q$ is logically equivalent to the *negation* of $P \Leftrightarrow Q$, $P \oplus Q \oplus R$ is logically equivalent to $P \Leftrightarrow Q \Leftrightarrow R$ itself. (One can omit all parentheses in logical expressions involving only $\oplus$ or $\Leftrightarrow$, since both connectives are commutative and associative.) The reason for this is that $\oplus$ is a mutual exclusivity connective, whereas $\Leftrightarrow$ is an identity connective. Hence, $P \oplus Q \oplus R$ is true precisely when $P \oplus Q$ and R have opposite truth values, which occurs precisely when $P \Leftrightarrow Q$ and R have identical truth values. Generalizing this pattern gives strings of propositions connected by $\oplus$ or $\Leftrightarrow$ that alternate in accordance with the following identities:

$$\text{(A)} \quad \bigoplus_{i=1}^{n} P_i \equiv \underset{i=1}{\overset{n}{\Leftrightarrow}} P_i, \quad \text{for } n \text{ odd;}$$

$$\text{(B)} \quad \bigoplus_{i=1}^{n} P_i \equiv \neg\left(\underset{i=1}{\overset{n}{\Leftrightarrow}} P_i \right), \quad \text{for } n \text{ even.}$$

We now prove these identities by mathematical induction on the number of propositions.

Proof. Basis: The logical equivalence $P_1 \oplus P_2 \equiv \neg(P_1 \Leftrightarrow P_2)$ follows directly from the truth tables for the two expressions.

Induction Step: Assume the identities true for an integer $n \geq 2$. We will show them true for $n + 1$.

(A) n is odd. We begin with $\oplus_{i=1}^{n+1} P_i$, which can be rewritten $(\oplus_{i=1}^{n} P_i) \oplus P_{n+1}$. By the basis, this is equivalent to $\neg((\oplus_{i=1}^{n} P_i) \Leftrightarrow P_{n+1})$. By the induction hypothesis, this is equivalent to $\neg((\Leftrightarrow_{i=1}^{n} P_i) \Leftrightarrow P_{n+1})$. This, in turn, is just $\neg(\Leftrightarrow_{i=1}^{n+1} P_i)$, which concludes the induction step for case (A) and with it the proof of case (A).

(B) n is even. We begin with $\oplus_{i=1}^{n+1} P_i$, which can be rewritten $(\oplus_{i=1}^{n} P_i) \oplus P_{n+1}$. By the basis, this is equivalent to $\neg((\oplus_{i=1}^{n} P_i) \Leftrightarrow P_{n+1})$. By the induction hypothesis, this is equivalent to $\neg(\neg(\Leftrightarrow_{i=1}^{n} P_i) \Leftrightarrow P_{n+1})$. Since $\neg(\neg P \Leftrightarrow Q)$ is true just when P and Q have identical truth values (i.e., $\neg(\neg P \Leftrightarrow Q) \equiv (P \Leftrightarrow Q)$), this in turn yields $(\Leftrightarrow_{i=1}^{n} P_i) \Leftrightarrow P_{n+1}$, which is just $\Leftrightarrow_{i=1}^{n+1} P_i$. This concludes the induction step for case (B) and with it the proof of case (B).

Acknowledgments. I would like to acknowledge the comments of an anonymous referee. I also wish to dedicate this note to Mr. Samuel Block, an outstanding teacher of mathematics, from whom I first learned to appreciate identities and their proofs.

Using Complex Solutions to Aid in Graphing

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The usual method of graphing $y = ax^2 + bx + c$ is to find the vertex by completing the square and find the x-intercepts by solving $ax^2 + bx + c = 0$. When the solutions to $ax^2 + bx + c = 0$ are complex, we say the graph does not cross the x-axis, and the complex solutions are of no further use to us. We will show that the complex solutions to $ax^2 + bx + c = 0$ are useful in graphing $y = ax^2 + bx + c$.

By completing the square we can write $y = ax^2 + bx + c$ as

$$y - \frac{4ac - b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

having vertex

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right).$$

And the solutions to $y = ax^2 + bx + c$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac$ is negative, then the solutions may be written

$$\frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i.$$

We notice the resemblance between the solutions and the vertex. Therefore if the solutions are $\alpha \pm \beta i$, then the vertex is at $(\alpha, a\beta^2)$. If the solutions are real and written in the form $\alpha \pm \beta$, then the vertex is at $(\alpha, -a\beta^2)$. The negative sign is needed because if $a < 0$, then the parabola is concave down, so the ordinate value of the vertex is positive or zero. And if $a > 0$, then the parabola is concave up so the ordinate value of the vertex is negative or zero.

PROBLEMS

LOREN C. LARSON, *editor*
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Texas Christian University

Proposals

To be considered for publication, solutions should be received by September 1, 1993.

1418. *Proposed by Irvin Roy Hentzel and Richard H. Sprague, Iowa State University, Ames, Iowa.*

Given three distances a, b, c, construct (using straightedge and compass and without analytic geometry) a square $ABCD$ and a point P such that $|PA| = a$, $|PB| = b$, and $|PC| = c$.

1419. *Proposed by William P. Wardlaw, U.S. Naval Academy, Annapolis, Maryland*

Show that for each odd prime p, there is an integer g such that $1 < g < p$ and g is a primitive root modulo p^n for every positive integer n.

1420. *Proposed by Cristian Turcu, London, England.*

If $\alpha, \beta, \gamma, \delta$ are real numbers, n is an odd integer, $\cos\alpha + \cos\beta + \cos\gamma + \cos\delta = 0$, and $\sin\alpha + \sin\beta + \sin\gamma + \sin\delta = 0$, prove that $\cos n\alpha + \cos n\beta + \cos n\gamma + \cos n\delta = 0$ and $\sin n\alpha + \sin n\beta + \sin n\gamma + \sin n\delta = 0$.

1421. *Proposed by Jiro Fukuta, Motosu-gun, Gifu-ken, Japan.*

If a polygon $A_1A_2 \dots A_n$ has an inscribed circle with center I and a circumcircle with center O, and C_i is the circumcenter of the triangle IA_iA_{i+1} $(i = 1, 2, \dots, n$, where $A_{n+1} = A_1)$, prove that the C_i's are concyclic.

ASSISTANT EDITORS: CLIFTON CORZAT, BRUCE HANSON, RICHARD KLEBER, KAY SMITH, *and* THEODORE VESSEY, *St. Olaf College and* MARK KRUSEMEYER, *Carleton College. We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren Larson, Department of Mathematics, St. Olaf College, 1520 St. Olaf Ave., Northfield, MN 55057-1098 or mailed electronically via fax: (507) 663-3549 or e-mail: larson@stolaf.edu.

1422. *Proposed by David Callan, University of Wisconsin, Whitewater, Wisconsin.*

Let A and B be $r \times n$ matrices with $r \leqslant n$ and rank $A = r$. Suppose that there is a nonzero constant k such that the determinant of every one of the $\binom{n}{r}$ r-square submatrices of B is k times the corresponding subdeterminant of A. Show that A and B have the same row space.

Quickies

Answers to the Quickies are on page 132.

Q802. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Prove that if one altitude intersects two other altitudes of a tetrahedron, then all four altitudes of the tetrahedron are concurrent.

Q803. *Proposed by Pierre Barnouin, Cabris, France.*

Determine a set of eight points in euclidean 3-space such that the squares of the distances from any of them to the others are seven distinct numbers in arithmetical progression. (Take as unit the shortest of these distances.)

Q804. *Proposed by Norman Schaumberger, Hofstra University, Hempstead, New York.*

It is known that if a, b, c are positive numbers then

$$a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}.$$

(See, for example, D. S. Mitrinović, *Elementary Ineqalities*, 1964, Ex. 3.38, p. 89, and Ex. 3.61, p. 90.)

Prove that

$$\left(\frac{a+b+c}{3}\right)^{a+b+c} \geq a^b b^c c^a.$$

Solutions

A Dynamical System Recursion — April 1992

1393. *Proposed by Florin S. Pîrvănescu, Slatina, Romania.*

The sequence (a_n) of real numbers is defined inductively by

$$a_1 = a \quad \text{and} \quad a_{n+1} = a_n^2 - 2 \quad \text{for } n \geq 1.$$

Compute the product $a_1 a_2 \ldots a_n$.

I. *Solution by Gordon Williams, Virginia Military Institute, Lexington, Virginia.*

Let a be any real number. Then for $a_1 = a$ and $a_{n+1} = a_n^2 - 2$, $n \geq 1$, and $P_n(a) = a_1 a_2 \dots a_n$, we have the following results:

(1) for $a < 0$, $P_n(a) = -P_n(-a)$;

(2) $P_n(0) = 0$ and $P_n(2) = 2^n$;

(3) for $0 < a < 2$, $a = 2\cos t$, where $0 < t < \pi/2$, $P_n(a) = \sin(2^n t)/\sin t$; and

(4) for $2 < a = 2\cosh t$, where $t > 0$, $P_n(a) = \sinh(2^n t)/\sinh t$.

The first result follows at once from $a_2 = (\pm a_1)^2 - 2$, and the second result is obvious. The third result follows from

$$a_n = 2\cos(2^n t) \quad \text{and} \tag{a}$$

$$2^n \sin t \cdot \cos t \cdot \cos 2t \cdot \cos 2^2 t \cdot \ \cdots \ \cdot \cos(2^{n-1}t) = \sin(2^n t). \tag{b}$$

Part (a) can be proved using induction by noting that

$$a_{n+1} = a_n^2 - 2 = 2\left(\cos^2(2^n t) - 1\right) = 2\cos(2 \cdot 2^n t)$$

and (b) follows easily by repeated use of the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

The fourth claim can be proved using the analogous identities for the hyperbolic functions.

II. *Solution by Heinz-Jürgen Seiffert, Berlin, Germany.*

If $a = 2$ then $a_n = 2$ for all n and therefore $a_1 a_2 \dots a_n = 2^n$. For $a \neq 2$, let α and β be the roots of the quadratic equation $x^2 - ax + 1 = 0$. Then we have $\alpha\beta = 1$, and $\alpha + \beta = a$. An easy induction yields

$$a_n = \alpha^{2^{n-1}} + \beta^{2^{n-1}}$$

and

$$a_1 a_2 \dots a_n = \frac{\alpha^{2^n} - \beta^{2^n}}{\alpha - \beta}.$$

III. *Solution by Reiner Martin, student, University of California at Los Angeles, Los Angeles, California.*

We claim that

$$a_n = 2T_{2^{n-1}}(a/2) \quad \text{and} \quad a_1 a_2 \dots a_n = \frac{2T_{2^n-1}(a/2) - 2T_{2^n+1}(a/2)}{4 - a^2},$$

where T_k denotes the Chebyshev polynomial of degree k. We use the facts that $T_1(x) = x$ and $2T_k T_l = T_{k+l} + T_{k-l}$.

Using induction, the first claim follows from $a_1 = 2T_1(a/2)$ and $2T_{2^n} = (2T_{2^{n-1}})^2 - 2$. Again using induction, the second claim follows from

$$a_1 = \frac{2T_1(a/2) - 2T_3(a/2)}{4 - a^2}$$

and

$$\frac{2T_{2^n-1}(a/2) - 2T_{2^n+1}(a/2)}{4 - a^2} \cdot a_{n+1} = \frac{2T_{2^{n+1}-1}(a/2) - 2T_{2^{n+1}+1}(a/2)}{4 - a^2},$$

which is a consequence of the first claim.

Also solved by David M. Bloom, David Callan, David Doster, Jiro Fukuta (Japan), Hans Kappus (Switzerland), H. K. Krishnapriyan, Kee-Wai Lau (Hong Kong), Peter W. Lindstrom, Beatriz Margolis (France), Andreas Müller (France), F. C. Rembis, Michael Tehranchi, Staffan Wrigge (Sweden), Michael Vowe (Switzerland), Yuching You, and the proposer.

Callan notes that further information about this sequence is contained in the solution to Problem E3036 *The American Mathematical Monthly*, October 1987, p. 789. *Williams* indicates that there is an extensive discussion of the mapping $f(x) = x^2 - 2$ in the article "Overview: Dynamics of Simple Maps," R. L. Devaney, *Chaos and Fractals*, which is Volume 39 of "Proceedings of Symposia in Applied Mathematics" of the American Mathematical Society.

A Combinatorial Identity **April 1992**

1394. *Proposed by David Callan, University of Wisconsin, Whitewater, Wisconsin.*

Prove the identity

$$\sum_{k=0}^{n} \frac{1}{2k+1}\binom{2n+1}{n-k} = \sum_{k=0}^{n} \frac{4^k}{2k+1}\binom{2n-2k}{n-k}.$$

Solution by Marc Noy, Universidad Politècnica de Catalunya, Barcelona, Spain.

Recently D. Zeilberger (see [1], [2]) has shown that most combinatorial identities can be proved using an efficient algorithm based on Gosper's algorithm for indefinite summation of hypergeometric functions: the present identity is no exception. Let

$$F(n,k) = \frac{1}{2k+1}\binom{2n+1}{n-k};$$

$$f(n) = \sum_{k=0}^{n} F(n,k).$$

Running Zeilberger's algorithm (which has been implemented in *Maple*) one gets the following. If you put

$$G(n,k) = -2\frac{(2k+1)(n+1)F(n,k)}{n+k+2}$$

then

$$(2n+3)F(n+1,k) - (8n+8)F(n,k) = G(n,k) - G(n,k-1),$$

an identity that can be proved routinely since it involves no summation. Now it is only a matter of summing the last equation from $k = 0$ to $n+1$ and cancelling terms in order to get a first order recurrence satisfied by the $f(n)$, namely

$$(2n+3)f(n+1) - (8n+8)f(n) = G(n,n+1) - G(n,-1) = 2\binom{2n+1}{n}.$$

Now do the same thing with the second expression, that is,

$$F'(n,k) = \frac{4k}{2k+1}\binom{2n-2k}{n-k};$$

$$f'(n) = \sum_{k=0}^{n} F'(n,k).$$

The corresponding identity is

$$(2n+3)F'(n+1,k) - (8n+8)F'(n,k) = G'(n,k) - G'(n,k-1),$$

with $G'(n,k) = -(8k+4)F'(n,k)$. It turns out that the same first order recurrence is satisfied by the $f'(n)$, and since $f(n)$ and $f'(n)$ agree for $n = 0$, they agree for every $n \geq 0$.

1. D. Zeilberger, The Method of creative telescoping, *Journal of Symbolic Computation*, 11 (1991), 195–204.
2. D. Zeilberger, "A Maple program for proving hypergeometric identities," *SIGSAM Bulletin*, 25 (1991), 4–13.

Also solved by Shalosh B. Ekhad, Rolf Richberg (Germany), Michael Vowe (Switzerland), and the proposer.

An n-gon Product Identity **April 1992**

1395. *Proposed by Ioan Sadoveanu, Ellensberg, Washington.*

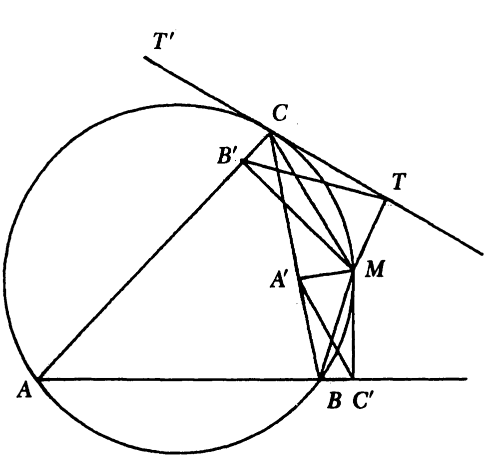
Let $A_1A_2\dots A_n$ be an n-gon circumscribing a circle, and let $B_1, B_2, \dots, B_n$ denote the points of tangency of the sides. Let M be a point on the circumference of the incircle. Show that

$$\prod_{i=1}^{n} d(M, B_{\sigma(i)}B_{\sigma(i+1)}) = \prod_{i=1}^{n} d(M, A_iA_{i+1})$$

for any permutation σ of $\{1, 2, \dots, n\}$. (Here, $B_{\sigma(n+1)} = B_{\sigma(1)}$ and $A_{n+1} = A_1$.)

I. *Solution by John G. Heuver, Grande Prairie Composite High School, Grande Prairie, Alberta, Canada.*

Consider ΔABC with M on the circumscribed circle and tangent line at C. Let MC', MB', MA', and MT be the perpendiculars to AB, BC, CA and the tangent at C, respectively. The quadrilaterals $A'BC'M$ and $TCB'M$ are cyclic and $\angle A'MC' = \angle ABC = \angle ACT' = \angle TMB'$. Also $\angle MCT = \angle MBA'$ and since $\angle MCT = \angle MB'T$ and $\angle MBA' = \angle MC'A'$, we conclude that $\Delta MA'C'$ is similar to $\Delta MTB'$. From this it follows that $MA' \cdot MB' = MC' \cdot MT$. For convenience, denote $MA' = p_1$, $MB' = p_2$, $MC' = p_3$ and $MT = t_3$. Then we have $p_1p_2 = p_3t_3$. For the tangent at B we obtain $p_1p_3 = p_2t_2$, and similarly for the tangent at A we have $p_2p_3 = p_1t_1$. Multiplying the three expressions and cancelling we obtain $p_1p_2p_3 = t_1t_2t_3$. This establishes the result for $n = 3$.



We obtain the proof for the general case by induction on n. Assume that for $n \geq 3$, $p_1p_2\dots p_n = t_1t_2\dots t_n$ and insert vertex $A_{r'}$ between A_r and A_{r+1}. Then we have a triangle with perpendicular distances $p_{r'}, p_{r'+1}, p_r$ to sides $A_rA_{r'}, A_{r'}A_{r+1}, A_rA_{r+1}$, respectively, and the perpendicular distance $t_{r'}$ to the tangent $A_{r'}$. We conclude that $p_{r'}p_{r'+1} = p_r t_{r'}$. This implies that $p_1 p_2 \cdots p_r p_{r'} p_{r'+1} p_{r+1} \cdots p_n = t_1 t_2 \cdots p_r t_{r'} t_{r+1} \cdots t_n$, or equivalently,

$$\underbrace{p_1 p_2 \cdots p_{r'} p_{r'+1} p_{r+1} \cdots p_n}_{n+1 \text{ terms}} = \underbrace{t_1 t_2 \cdots t_{r'} t_{r+1} \cdots t_n}_{n+1 \text{ terms}}.$$

Hence the statement holds for all $n \geq 3$. Since the choice of A_r and A_{r+1} was arbitrary, it follows that

$$\prod_{i=1}^{n} d(M, B_{\sigma(i)} B_{\sigma(i+1)}) = \prod_{i=1}^{n} d(M, A_i A_{i+1})$$

for any permutation σ of $\{1, 2, \ldots, n\}$.

II. *Solution by Reiner Martin, student, University of California at Los Angeles, Los Angeles, California.*

Let C be the center of the circle and r its radius. Denote the angle $\angle MCB_i$ by α_i and the tangent at B_i by T_i. Simple geometric considerations show that $d(M, T_i) = r(1 - \cos\alpha_i)$ and that $d(M, B_i B_j) = r(\cos((\alpha_i - \alpha_j)/2) - \cos((\alpha_i + \alpha_j)/2))$.

But

$$(1 - \cos\alpha_i)(1 - \cos\alpha_j) = \left(\cos\frac{\alpha_i - \alpha_j}{2} - \cos\frac{\alpha_i + \alpha_j}{2}\right)^2.$$

Therefore,

$$\prod_{i=1}^{n} d(M, B_{\sigma(i)} B_{\sigma(i+1)}) = \left(\prod_{i=1}^{n} d(M, T_{\sigma(i)}) d(M, T_{\sigma(i+1)})\right)^{1/2}$$
$$= \prod_{i=1}^{n} d(M, T_i) = \prod_{i=1}^{n} d(M, A_i A_{i+1}).$$

Also solved by Jiro Fukuta (Japan), Hans Kappus (Switzerland), Serif Kapudere (student, Turkey), and the proposer.

A Triangle Dissection **April 1992**

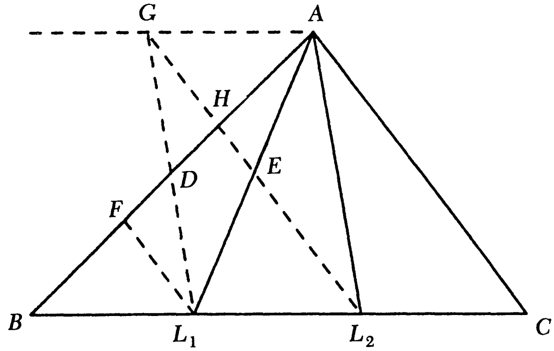
1396. *Proposed by Jiro Fukuta, Motosu-gun, Gifu-ken, Japan.*

Let ABC be an arbitrary triangle, let L_1 and L_2 be trisection points of BC, arranged in order from B to C. Describe a method for dissecting triangle ABL_1 into four parts, each of which is a triangle or a quadrilateral, so that the parts can be reassembled to form a triangle congruent to triangle AL_2C.

I. *Solution by R. S. Tiberio, Natick, Massachusetts.*

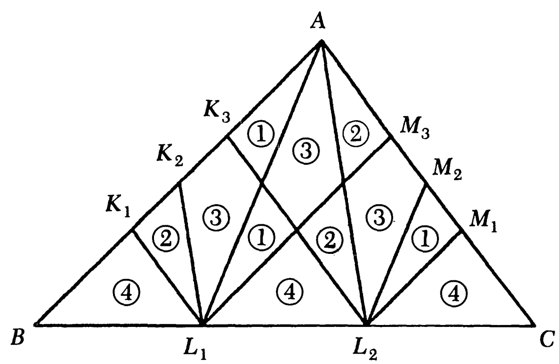
Through L_1 draw a line parallel to AC intersecting AB at F. Let D be the midpoint of AB. Draw a line through L_1 and D intersecting the line through A parallel to BC at point G. Draw GL_2 intersecting AL_1 at E and AB at H. The four pieces are ΔAHE, ΔDFL_1, ΔBFL_1, and quadrilateral $DHEL_1$.

The reassembly is as follows. With point D as center, rotate ΔBDL_1 through 180° (point B ends up coincident with point A). Now rotate ΔGEL_1 through 180° using point E as center (point G ends up on L_2). The four original pieces are now lying on ΔGAL_2. It is an easy exercise to show that this triangle is congruent to ΔCL_2A.



II. *Solution by Serif Kapudere, student, Bilkent University, Ankara, Turkey.*

Let K_1 and K_3 be the trisection points of BA, arranged in order from B to A, let M_1 and M_3 be the trisection points of CA, arranged in order from C to A, and let K_2 and M_2 bisect AB and AC respectively. Then $M_3L_1 \| AB$, $M_2L_2 \| AL_1$, $M_1L_2 \| AB$, $L_1K_1 \| AC$, $L_1K_2 \| AL_2$, and $L_2K_3 \| AC$. It is now an easy matter to show that the triangles and quadrilaterals with the same labels (see following figure) are congruent.



Also solved by María Ascensión López Chamorro (Spain), Ralph Merrill, Irene and Kao Hwa Sze, and the proposer.

Average Number of Transpositions **April 1992**

1397. *Proposed by John O. Kiltinen, Northern Michigan University, Marquette, Michigan.*

It is well known that every permutation on a finite set can be expressed as a "product" of transpositions. For each permutation σ of $\{1, 2, \ldots, n\}$ let $F_n(\sigma)$ denote the minimum number of transpositions needed to represent σ as a product. Find the average value of F_n over the set of all the permutations of $\{1, 2, \ldots, n\}$.

I. *Solution by Paul J. Zwier, Calvin College, Grand Rapids, Michigan.*

To simplify matters, we define $F_n(\varepsilon) = 0$, where ε is the identity permutation. We shall show that the average value, $\overline{F}_n$, is

$$\overline{F}_n = \sum_{k=2}^{n} \frac{k-1}{k} = n - H(n),$$

where $H(n)$ is the sum of the first n terms of the harmonic series.

Note that if σ is a permutation and if $(i)\sigma = j$ where $i \neq j$, then there is a minimal representation of σ of the form $t_1 t_2 \cdots t_{F_n(\sigma)-1}(ij)$. To see this, take a minimal representation of σ and use the identities

$$\begin{aligned}(ab)(ab) &= \varepsilon\\ (ab)(ac) &= (bc)(ab)\\ (ab)(cd) &= (cd)(ab)\\ (ab)(bc) &= (bc)(ac)\end{aligned}$$

to drive the left-most i to the right until it appears only in the right-most transposition. The fact that the representation does not collapse in the process follows from the fact that the original representation is minimal. Since $(i)\sigma = j$, it follows that the right-most transposition is (ij).

For each $1 \le i \le n$, let $T_i = \{\sigma \in S_n\colon (1)\sigma = i\}$. For $i \neq 1$, let $\phi\colon T_1 \to T_i$ be defined by $(\sigma)\phi = \sigma(1i)$. It is easy to see that ϕ is one-to-one and onto and also that if $\sigma = t_1 t_2 \cdots t_{F_n(\sigma)}$ is a minimal representation of σ as the product of transpositions, $(\sigma)\phi$ has the minimal representation $t_1 t_2 \cdots t_{F_n(\sigma)}(1i)$ and thus $F_n((\sigma)\phi) = F_n(\sigma) + 1$.

We can now find a recursive formula for $\overline{F}_n$. Certainly $\overline{F}_2 = \frac{1}{2}$. For $n > 2$,

$$\begin{aligned} n!\overline{F}_n &= \sum_{i=1}^{n} \sum_{\sigma \in T_i} F_n(\sigma)\\ &= (n-1)!\overline{F}_{n-1} + (n-1)(n-1)!\overline{F}_{n-1} + (n-1)(n-1)!\\ &= n!\overline{F}_{n-1} + (n-1)(n-1)! \end{aligned}$$

From this it follows that

$$\overline{F}_n = \overline{F}_{n-1} + \frac{n-1}{n}.$$

Using the fact that $\overline{F}_2 = \frac{1}{2}$ we find that

$$\overline{F}_n = \sum_{n=2}^{n} \frac{k-1}{k} = n - H(n).$$

II. *Solution by Michael H. Andreoli, Miami Dade Community College (North), Miami, Florida.*

We use the fact that any permutation of n elements can be uniquely represented as a product of disjoint cycles.

Since $(1, 2, \ldots, m) = (1, 2)(1, 3) \ldots (1, m)$ it is clear that any m-cycle can be represented by a product of $m - 1$ transpositions. Moreover, it is impossible to represent an m-cycle by fewer than $m - 1$ transpositions. A short proof of this fact can be found in the solution (by O. P. Lossers) of Problem E3058 in *The American Mathematical Monthly*, Vol. 93, December 1986, p. 820.

Let $A(n)$ denote the average value of F over S_n, and let $A(n, k)$ denote the average value of F over $T_k = \{\tau \in S_n\colon$ The cycle in τ containing "n" has k elements$\}$. By the result quoted in the preceding paragraph, we have $A(n, k) = (k-1) + A(n-k)$.

The number of permutations of $\{1, 2, \ldots, n\}$ for which the cycle containing "n" has $k \ge 1$ elements is $\binom{n-1}{k-1}(k-1)!(n-k)! = (n-1)!$. Thus, the size of T_k is independent of k, and we have

$$A(n)=\frac{1}{n}\sum_{k=1}^{n}A(n,k)=\frac{1}{n}\sum_{k=1}^{n}((k-1)+A(n-k))=\frac{n-1}{2}+\frac{1}{n}\sum_{k=1}^{n}A(n-k).$$

Taking $A(0)=0$, and solving by iteration yields $A(1)=0$, and for $n\geq 2$, $A(n)=1/2+2/3+3/4+\cdots+(n-1)/n$.

Also solved by David M. Bloom, David Callan, Stephen R. Cavior, Jane Friedman and Marquis Griffith and Ryan Jackson and Mika Wheeler, Russell Jay Hendel, R. High, Peter W. Lindstrom, Reiner Martin (student), José Heber Nieto (Venezuela), Frederic Schoenberg (student), William P. Wardlaw, and the proposer.

Bloom suggests the following related problem: If $g(\sigma)$ denotes the number of orbits of the permutation $\sigma\in S_n$, prove that $\Sigma_{\sigma\in S_n}g(\sigma)$ equals the number of elements of S_{n+1} that have exactly two orbits.

Answers

Solutions to the Quickies on page 125.

A802. Let the vertices of the tetrahedron be A_0, A_1, A_2, A_3, and assume the altitude from A_0 intersects the altitudes from A_1 and A_2. We now choose a vector origin to be the point P that is the intersection of the altitude from A_0 with the altitude from A_1. The vector $\mathbf{A}_i$, $i=1,2,3,4$, will denote the vector from P to A_i. Since $\mathbf{A}_0$ is orthogonal to the face opposite A_0, we immediately have

$$\mathbf{A}_0\cdot\mathbf{A}_1=\mathbf{A}_0\cdot\mathbf{A}_2=\mathbf{A}_0\cdot\mathbf{A}_3, \tag{1}$$

and similarly

$$\mathbf{A}_1\cdot\mathbf{A}_0=\mathbf{A}_1\cdot\mathbf{A}_2=\mathbf{A}_1\cdot\mathbf{A}_3. \tag{2}$$

The intersection of the altitudes from A_0 and A_2 is $k\mathbf{A}_0$ for some k. Then as above, we also have

$$(\mathbf{A}_2-k\mathbf{A}_0)\cdot\mathbf{A}_0=(\mathbf{A}_2-k\mathbf{A}_0)\cdot\mathbf{A}_1=(\mathbf{A}_2-k\mathbf{A}_0)\cdot\mathbf{A}_3.$$

Then using (1) and (2), we have that all the $\mathbf{A}_i\cdot\mathbf{A}_j$ $(i\neq j)$ are equal. Hence P is the orthocenter of the tetrahedron.

Proceeding in the same way, it follows that if one altitude of an n-dimensional simplex intersects $n-1$ other altitudes, all $n+1$ altitudes are concurrent.

This problem first appeared as Problem E2226 in *The American Mathematical Monthly*. This solution is much simpler than the one published there (February 1971, Vol. 78, No. 2, p. 200).

A803. Such a set of eight points are the vertices of a right-angled parallelpiped measuring 1 by $\sqrt{2}$ by 2.

A804. Since $b/(a+b+c)+c/(a+b+c)+a/(a+b+c)=1$, the weighted arithmetic mean—geometric mean inequality gives

$$\begin{aligned}a+b+c&=\frac{(a+b+c)^2}{a+b+c}=\frac{a^2+b^2+c^2+2ab+2bc+2ca}{a+b+c}\\&\geq 3\left(\frac{ba}{a+b+c}+\frac{cb}{a+b+c}+\frac{ac}{a+b+c}\right)\\&\geq 3\left(a^{b/a+b+c}b^{c/a+b+c}c^{a/a+b+c}\right).\end{aligned}$$

Comments

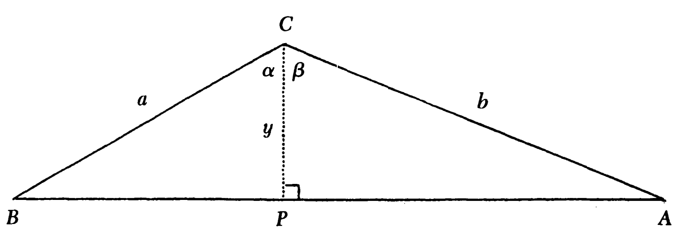
1375. *David Moews, student, University of California, Berkeley,* used a computer search to find that

$$\{155213816675809492328855458243, 710010917562137, 52978181, 16501, 67, 17, 7, 5, 4, 3\}$$

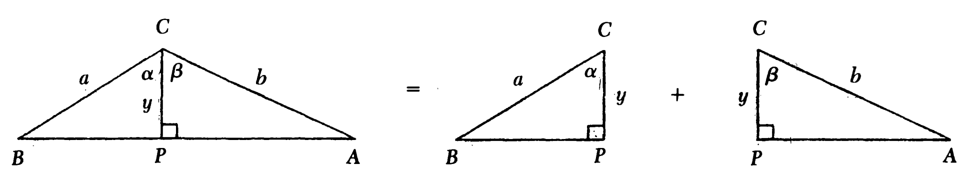
is a set satisfying the conditions of the problem. Thus there exists such a set not including 2. This was heretofore an open problem.

Proof without Words:

Sin$(\alpha + \beta)$ for $\alpha + \beta < \pi$



$$y = a \cos \alpha = b \cos \beta$$



$$\tfrac{1}{2}ab \sin(\alpha + \beta) = \tfrac{1}{2}ay \sin \alpha + \tfrac{1}{2}by \sin \beta$$

$$= \tfrac{1}{2}ab \cos \beta \sin \alpha + \tfrac{1}{2}ba \cos \alpha \sin \beta$$

$$\therefore \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

—CHRISTOPHER BRUENINGSEN
THE KISKI SCHOOL
SALTSBURG, PA 15681

Comments

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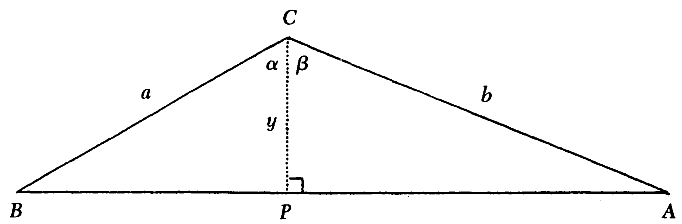
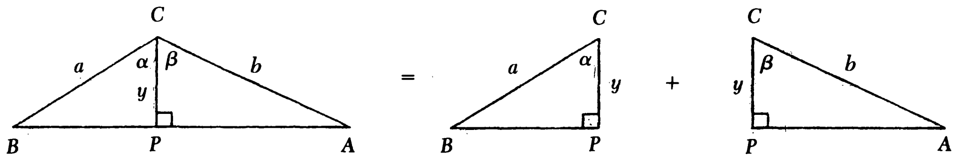
Proof without Words:

$\text{Sin}(\alpha+\beta)$ for $\alpha+\beta<\pi$



$$y = a\cos\alpha = b\cos\beta$$



$$\tfrac{1}{2}ab\sin(\alpha+\beta) = \tfrac{1}{2}ay\sin\alpha + \tfrac{1}{2}by\sin\beta$$

$$= \tfrac{1}{2}ab\cos\beta\sin\alpha + \tfrac{1}{2}ba\cos\alpha\sin\beta$$

$$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

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REVIEWS

PAUL J. CAMPBELL, *editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Schaufele, Christopher, and Nancy Zumoff, *Earth Algebra: College Algebra with Applications to Environmental Issues*, preliminary version, HarperCollins, 1993; xx + 364 pp, \$20. ISBN 0-06-500887-1

In the past few years, much money and effort has gone into thinking about how to reform calculus in college, while high-school teachers have grappled with how to implement the NCTM *Standards* in their courses. What about college algebra, that course between two worlds? This book represents a major and welcome initiative to turn college algebra into a course relevant to student concerns, in this case about the environment. Another innovative feature is focusing student tasks on working in small groups (using graphing calculators) and presenting written and oral reports. "Each mathematics topic presented is used in an application; there are no extraneous topics." Mathematical topics treated are functions (linear, quadratic, logarithmic, exponential, piecewise, composite, inverse), systems of equations and matrices, geometric series, and linear inequalities and linear programming. This is a short but ambitious list of topics, and each is treated in far less detail than in other contemporary college algebra books; perhaps because the book is not in its 5th edition, it does not feature the endless repetitive examples that characterize many others. The book does not intend to develop the facility with algebra and trigonometry that is important to success in a traditional calculus course; it is not a pre-(traditional)calculus book. Much of the book centers on "modeling," characterized as fitting lines and curves to (real) data by exact fitting at two or three chosen points. There is no mention of least-squares, functional transforms of data, or error bounds for predictions, nor of using graphing calculators for such tasks. A third of the book is a supplement of reprints from graphing-calculator manuals, leaving a scant 200 pp (with generous white space) of text. Well, chuck the reprints; a manual comes with the graphing calculator that a student buys, so why repeat the material here (together with instructions for calculators that the student didn't buy)? Besides, the supplement may not remain current as new models appear. A "first" edition of the book will appear in 1995, incorporating additional ideas based on the experience of users of the book. (What really makes this version "preliminary," from the publisher's point of view, is that it is just like a "real" book, printed in black and white instead of the multiple colors that are regarded today as essential to student learning.)

Skiena, Steven, *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*, Addison-Wesley, 1990; x + 334 pp, \$39.95. ISBN 0-201-50943-1

Version 2 of Mathematica incorporates "Combinatorica," a collection of algorithms for discrete mathematics, whose details are given in this book (users of Version 1.2 can get the code on diskette for \$15). In addition to algorithms to represent, generate, and investigate properties of graphs, the package also does permutations and combinations, partitions, compositions, and Young tableaux. The book shows the code for each algorithm and the results of applying it to an example, together with definitions, commentary on the example, and observations on the algorithm. Should anyone who works with discrete structures—or is trying to learn about them—be without a tool like this?

Koza, John R., *Genetic Programming: On the Programming of Computers by Means of Natural Selection*, MIT Press, 1992; xiv + 819 pp. ISBN 0–262–11170–5. Koza, John R., and James P. Rice, *Genetic Programming: The Movie*, 60-minute videotape, 1992; VHS NTSC format $34.95. ISBN 0–262–61084–1. VHS PAL or VHS SECAM format, $44.95.

What a marvelous idea: to get a computer to solve a problem without expressly programming it to! And it works, as this book and video demonstrate. The idea carries genetic algorithms one big step further. A genetic algorithm (e.g., for the traveling sales representative problem) starts from a collection of feasible solutions and "breeds" successive generations of solutions via genetic recombination and survival of the fittest, trying to arrive at an optimal solution. The idea of this book is that for many types of problems, you can start from a collection of randomly generated *computer programs*, evolve successive generations, and arrive at a program that will solve that *type* of problem. Examples of problem types: optimal control (backing a trailer truck), game-playing, robot motion-planning, regression, classification, solving equations, grammar induction. The book provides Lisp code for a simple version of genetic programming.

Ralston, Anthony, and Edwin D. Reilly, *Encyclopedia of Computer Science*, 3rd ed., Van Nostrand Reinhold, 1993; xxvi + 1558 pp, $150. ISBN 0–442–27679–6

Slightly larger print and a larger page size balance out; the 3rd edition of this valuable reference book actually has fewer pages than the 2nd edition, which appeared long ago—in 1983. How can this be? About 20% of the articles in the 2nd edition have been deleted or merged with others. Many articles appear largely without change, but more than 40% are new or extensively revised.

Wallich, Paul, Crunching epsilon: Cryptography may be the key to checking enormous proofs, *Scientific American* (January 1993) 130–140.

Holographic proofs promise the potential for probabilistically verifying mathematical proofs and computer programs. The idea is to convert an ordinary proof to logical form, produce a polynomial from that, and list values that the polynomial should have for different combinations of values of the variables. Verifying involves comparing randomly chosen asserted values of the polynomial with calculated values. Whether such proofs can ever be practical is debatable, because the logical form of a proof tends to be longer than the proof itself; if the proof has length N, the logical form has length $\mathcal{O}(N^{1+\epsilon})$, with a large constant and an ϵ between 0 and 1.

Peterson, Ivars, Monte Carlo physics: A cautionary lesson, *Science News* 142 (18 & 26 December 1992) 422. Ferrenberg, Alan M., D.P. Landau, and Y.Joanna Wong, Monte Carlo simulations: Hidden errors from "good" random number generators, *Physical Review Letters* 69 (23) (7 December1992) 3382–3384.

Physicists have sounded the alert that even some "high-quality" random-number generators can lead to biased simulation results. In particular, the new "subtract with borrow" random-number generator devised by George Marsaglia and Arif Zaman (Florida State University) consistently led to wrong results for a problem whose exact answer is known. In a related discovery, Shu Tezuka (IBM Tokyo Research Laboratory) and Pierre L'Ecuyer (University of Montréal in Québec) have proved that that random-number generator is "essentially equivalent" to linear congruential random-number generators, whose features (and disadvantages) have been analyzed extensively. Conclude the physicists: "[A] specific algorithm must be tested together with the random-number generator being used *regardless* of the tests which the generator has passed."

NEWS AND LETTERS

53rd ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

These solutions have been compiled and prepared by Loren Larson, St. Olaf College.

A-1. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions.

(i) $f(f(n)) = n$, for all integers n;

(ii) $f(f(n+2)+2) = n$ for all integers n;

(iii) $f(0) = 1$.

Solution. If $f(n) = 1-n$, then $f(f(n)) = f(1-n) = 1-(1-n) = n$, so (i) holds. Similarly, $f(f(n+2)+2) = f((-n-1)+2) = f(1-n) = n$, so (ii) holds. Clearly (iii) holds, and so $f(n) = 1-n$ satisfies the conditions.

Conversely, suppose f satisfies the three given conditions. From (ii), $f(f(f(n+2)+2)) = f(n)$, and applying (i) yields $f(n+2)+2 = f(n)$ or $f(n+2) = f(n) - 2$. An easy induction yields

$$f(n) = \begin{cases} f(0) - n & \text{if } n \text{ is even,} \\ f(1) + 1 - n & \text{if } n \text{ is odd.} \end{cases}$$

If $f(0) = 1$, then $f(1) = 0$ by (i), and therefore, $f(n) = 1 - n$.

A-2. Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series expansion about $x = 0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy .$$

Solution. From the binomial series, we see that

$$C(-y-1) = \frac{(y+1)(y+2)\cdots(y+1992)}{1992!}.$$

Therefore,

$$C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} = \frac{d}{dy}\left(\frac{(y+1)(y+2)\cdots(y+1992)}{1992!} \right).$$

Hence the integral in question is

$$\int_0^1 \frac{d}{dy}\left(\frac{(y+1)(y+2)\cdots(y+1992)}{1992!} \right) dy = \left. \frac{(y+1)(y+2)\cdots(y+1992)}{1992!} \right|_0^1 = 1993 - 1 = 1992.$$

A-3. For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy

$$(x^2+y^2)^m = (xy)^n.$$

Solution. There are no solutions if m is odd. If m is even, the only solution is $(n, x, y) = (m+1, 2^{m/2}, 2^{m/2})$.

If (n, x, y) is a solution, then by the arithmetic-mean – geometric-mean inequality, $(xy)^n = (x^2+y^2)^m \geq (2xy)^m$, so $n > m$. Let p be a prime number. Let a and b be the largest powers of p that divide x and y, respectively. Then the largest power of p dividing $(xy)^n$ is $(a+b)n$. If $a < b$, the largest power of p dividing $(x^2+y^2)^m$ is $2am$. But this implies that $(a+b)n = 2am$, and this contradicts $n > m$. Similarly, the assumption $a > b$ leads to a contradiction. Therefore $a = b$ for all primes p, and we conclude that $x = y$. Thus, the equation reduces to $(2x^2)^m = x^{2n}$, or equivalently, $x^{2(n-m)} = 2^m$. It follows that x is a positive power of 2, say 2^a. This implies $2(n-m)a = m$, or, $2an = (2a+1)m$. Since

$$\gcd(m, n) = \gcd(2a, 2a+1) = 1,$$

we must have $m = 2a$ and $n = 2a + 1$. Thus, m is necessarily even and the solution follows as claimed.

A-4. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \qquad n = 1, 2, 3, \ldots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \ldots$.

Solution. We will show that

$$f^{(k)}(0) = \begin{cases} (-1)^{k/2}k! & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

First we note that if $h(x)$ is a differentiable function and $x_1, x_2, \ldots,$ is a sequence strictly decreasing to 0 such that $h(x_n) = 0$, then by Rolle's Theorem, there exists a sequence $y_1, y_2, \ldots,$ strictly decreasing to 0, such that $h'(y_n) = 0$ $(x_{n+1} < y_n < x_n)$.

Now let $g(x) = f(x) - \dfrac{1}{1+x^2}$. Then $g\left(\dfrac{1}{n}\right) = 0$ for $n = 1, 2, \ldots$. Applying the result of the preceding paragraph to $g, g', g'', \ldots$ and invoking the continuity of $g^{(k)}$ at 0, we see that $g^{(k)}(0) = 0$ for $k = 0, 1, 2, 3, \ldots$. Thus, $f^{(k)}(0) = \dfrac{d^k}{dx^k}\left(\dfrac{1}{1+x^2}\right)\Bigg|_{x=0}$.

The Maclaurin series for $\dfrac{1}{1+x^2}$ is $\sum_{k=0}^{\infty}(-1)^k x^{2k}$, and hence $f^{(k)}(0)$ is equal to the values given above.

A-5. For each positive integer n, let $a_n = 0$ (or 1), if the number of 1's in the binary representation of n is even (or odd), respectively. Show that there do not exist positive integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j},$$

for $0 \le j \le m - 1$.

Solution. Observe that $a_{2n} = a_n$ and $a_{2n+1} = 1 - a_{2n} = 1 - a_n$.

Suppose that there exist k, m as above, and we may assume m is minimal for such.

Suppose first that m is odd. We'll suppose $a_k = a_{k+m} = a_{k+2m} = 0$, as it will be clear that the case $a_k = 1$ can be treated similarly. Since either k or $k + m$ is even, $a_{k+1} = a_{k+m+1} = a_{k+2m+1} = 1$. Again, since either $k + 1$ or $k + m + 1$ is even, $a_{k+2} = a_{k+m+2} = a_{k+2m+2} = 0$. By this means, we see that the terms $a_k, a_{k+1}, a_{k+2}, \ldots, a_{k+m-1}$ alternate between 0 and 1. Then since $m - 1$ is even, $a_{k+m-1} = a_{k+2m-1} = a_{k+3m-1} = 0$. But, since either $k + m - 1$ or $k + 2m - 1$ is even, that would imply that $a_{k+m} = a_{k+2m} = 1$, a contradiction.

Thus, m must be even. Extracting the terms with even indices in

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j},$$

for $0 \le j \le m - 1$, and using the fact that $a_r = a_{r/2}$ for even r, we get

$$a_{\lceil k/2 \rceil + i} = a_{\lceil k/2 \rceil + (m/2) + i} = a_{\lceil k/2 \rceil + m + i},$$

for $0 \le i \le (m/2) - 1$. (The even numbers $\ge k$ are $2\lceil k/2 \rceil$, $2\lceil k/2 \rceil + 2, \ldots$.) This contradicts the minimality of m.

Hence, there are no such k and m.

A-6. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

Solution. Given three points P, Q, R on the sphere, we consider where on the sphere a fourth point S must be located so that the tetrahedron $PQRS$ includes the center O. Let P', Q', R' be opposite on the sphere to P, Q, R respectively. We observe that O is in the interior of tetrahedron $PQRS$

if and only if S is in the interior of the solid angle whose edges are the half-lines OP', OQ', OR' extended. This is the case if and only if S lies within the spherical triangle with vertices P', Q', R'.

Thus, if P, Q, R are already chosen, then the probability that a random S gives a desirable tetrahedron is equal to the area of the spherical triangle PQR divided by the area of the sphere (since triangles PQR and $P'Q'R'$ are congruent).

It follows that the desired probability is the expected area of PQR divided by the area of the sphere, where P, Q, R are randomly chosen. To determine this quantity, we note that the expected area of PQR is equal to the expected area of $P'QR$. The sum of these two quantities is the expected area of a random sector of the circle with poles P, P'. This random sector subtends a random angle (as measured along the equator) between 0 and π. Its expected area is then $1/2$ that of a hemisphere or $1/4$ that of the sphere. Hence the expected area of PQR is $1/8$ that of the sphere, and therefore, the desired probability is $1/8$.

B-1. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of elements in A_S?

Solution. The smallest possible number of elements in A_S is $2n-3$.

Let $x_1 < x_2 < \cdots < x_n$ represent the elements of S. Then

$$\frac{x_1+x_2}{2} < \frac{x_1+x_3}{2} < \cdots < \frac{x_1+x_n}{2}$$

$$< \frac{x_2+x_n}{2} < \frac{x_3+x_n}{2} < \cdots < \frac{x_{n-1}+x_n}{2}$$

represent $(n-1)+(n-2) = 2n-3$ distinct elements of A_S, so A_S has at least $2n-3$ distinct elements.

If we take $S = \{1, 2, \ldots, n\}$, the elements of A_S are $\frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \ldots, \frac{2n-1}{2}$. There are only $(2n-1)-2 = 2n-3$ such numbers; thus there is a set A_S with at most $2n-3$ distinct elements. This completes the proof.

B-2. For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{n} \binom{n}{j}\binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, and $\binom{a}{b} = 0$ otherwise.)

Solution. We have

$$\begin{aligned}\sum_{k\geq 0} Q(n,k)x^k &= (1+x+x^2+x^3)^n \\ &= (1+x^2)^n(1+x)^n \\ &= \sum_{j\geq 0}\binom{n}{j}x^{2j}\sum_{i\geq 0}\binom{n}{i}x^i \\ &= \sum_{j\geq 0}\sum_{i\geq 0}x^{2j+i}\binom{n}{j}\binom{n}{i} \\ &= \sum_{k\geq 0}x^k\sum_{j\geq 0}\binom{n}{j}\binom{n}{k-2j}.\end{aligned}$$

Comparing coefficients of x^k, we derive the desired result.

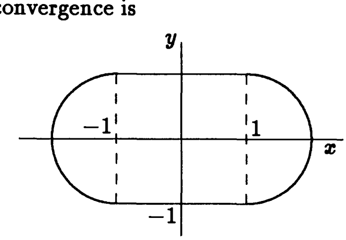
B-3. For any pair (x,y) of real numbers, a sequence $\left(a_n(x,y)\right)_{n\geq 0}$ is defined as follows:

$$a_0(x,y) = x,$$
$$a_{n+1}(x,y) = \frac{(a_n(x,y))^2 + y^2}{2}, \quad \text{for } n \geq 0.$$

Find the area of the region

$$\left\{(x,y) \;\middle|\; \left(a_n(x,y)\right)_{n\geq 0} \text{ converges}\right\}.$$

Solution. The area is $4+\pi$. The region of convergence is



namely, a (closed) square of side 2, $\{(x,y) \mid -1 \leq x, y \leq 1\}$ with (closed) semicircles of radius 1 centered at $(\pm 1, 0)$ described on two opposite sides.

If $\lim_{n\to\infty} a_n(x,y) = L$, then L must satisfy $L = \dfrac{L^2+y^2}{2}$; that is, L must be a root of the equation

$$r^2 - 2r + y^2 = 0. \qquad (1)$$

In such case, the equation must have real roots, so the discriminant, $4-4y^2$, must be nonnegative. Thus, a necessary condition for $\big(a_n(x,y)\big)$ to converge is that $|y| \le 1$.

Fix $|y| \le 1$. The roots of (1) are then $1-\sqrt{1-y^2}$ and $1+\sqrt{1-y^2}$, which are real and nonnegative. As $a_1(-x,y) = a_1(x,y)$, the interval of convergence is symmetric about $x = 0$. We shall assume then that $x \ge 0$; thus, $a_n(x,y) \ge 0$, for all n.

If $r_0 = 1 \pm \sqrt{1-y^2}$, then $a_{n+1}(x,y)$ is less than, equal to, or greater than r_0 according to whether $a_n(x,y)$ is less than, equal to, or greater than r_0 $\left(= (r_0^2+y^2)/2\right)$.

If $a_n(x,y)$ lies in the closed interval $\left[1-\sqrt{1-y^2},\, 1+\sqrt{1-y^2}\,\right]$, that is, between the roots of (1), then

$$a_n(x,y)^2 - 2a_n(x,y) + y^2 \le 0,$$

so that

$$1-\sqrt{1-y^2} \;\le\; a_{n+1}(x,y) \;\le\; a_n(x,y).$$

It follows that $\big(a_n(x,y)\big)_{n\ge 0}$ converges if x is in the closed interval

$$\left[1-\sqrt{1-y^2},\, 1+\sqrt{1-y^2}\,\right].$$

If $a_n(x,y)$ does not lie in the interval $\left[1-\sqrt{1-y^2},\, 1+\sqrt{1-y^2}\,\right]$, then

$$a_n(x,y)^2 - 2a_n(x,y) + y^2 > 0,$$

so that

$$a_{n+1}(x,y) > a_n(x,y).$$

Thus, if x, and therefore all $a_n(x,y)$, are greater than $1+\sqrt{1-y^2}$, then the sequence diverges. On the other hand, if x, and therefore all $a_n(x,y)$, lie between 0 and $1-\sqrt{1-y^2}$, the sequence converges monotonically to $1-\sqrt{1-y^2}$.

To summarize, $\big(a_n(x,y)\big)_{n\ge 0}$ converges if and only if

$$-1 \le y \le 1$$

and

$$-\left(1+\sqrt{1-y^2}\right) \;\le\; x \;\le\; 1+\sqrt{1-y^2}.$$

B-4. Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3-x}\right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

Solution. The smallest possible degree of $f(x)$ is 3984.

By the Division Algorithm, we can write $p(x) = (x^3-x)q(x) + r(x)$, where $q(x)$ and $r(x)$ are polynomials, the degree of $r(x)$ is less than 3, and the degree of $q(x)$ is less than 1989. Then

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3-x}\right) = \frac{d^{1992}}{dx^{1992}}\left(\frac{r(x)}{x^3-x}\right).$$

Because $p(x)$ and x^3-x have no nonconstant common factor, neither do $r(x)$ and x^3-x.

So let $p_0(x)$ be a polynomial relatively prime to x^3-x and of degree $n < 3$. We will show, by induction, that

$$\frac{d^k}{dx^k}\left(\frac{p_0(x)}{x^3-x}\right) = \frac{p_k(x)}{(x^3-x)^{k+1}},$$

where $p_k(x)$ is a polynomial relatively prime to x^3-x and $\deg(p_k) = \deg(p_0) + 2k < 3(k+1)$. We have

$$\begin{aligned}\frac{d}{dx}&\left(\frac{p_k(x)}{(x^3-x)^{k+1}}\right)\\ &= \frac{(x^3-x)p_k'(x) - (k+1)(3x^2-1)p_k(x)}{(x^3-x)^{k+2}}\\ &= \frac{p_{k+1}(x)}{(x^3-x)^{k+2}}.\end{aligned}$$

The numerator, $p_{k+1}(x)$, is relatively prime to x^3-x, since x^3-x divides the left term and is relatively prime to the right term. If $p_k(x) = ax^n + \textit{lower order terms}$, $a \ne 0$, then the leading coefficient of $p_{k+1}(x)$ is $a(n-3(k+1))$, and since $n \ne 3(k+1)$,

$$\begin{aligned}\deg(p_{k+1}) &= \deg(p_k) + 2\\ &= \deg(p_0) + 2k + 2\\ &< 3 + 3k + 2 < 3(k+2).\end{aligned}$$

This completes the induction.

Thus,

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{r(x)}{x^3-x}\right) = \frac{f(x)}{g(x)},$$

where $\deg(f) = \deg(r) + 2(1992)) \ge 3984$. The minimum is attained when $r(x)$ is a constant.

B-5. Let D_n denote the value of the $(n-1)\times(n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}$$

Is the set $\left\{\frac{D_n}{n!}\right\}_{n\geq 2}$ bounded?

Solution. The set $\left\{\frac{D_n}{n!}\right\}_{n\geq 2}$ forms a sequence which strictly increases to infinity; it is therefore unbounded.

Using properties of determinants,

$$D_n = \begin{vmatrix} 3 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 4 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & n & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & n+1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 & 0 & \cdots & 0 & 0 & -n \\ 0 & 3 & 0 & \cdots & 0 & 0 & -n \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-1 & -n \\ 1 & 1 & 1 & \cdots & 1 & 1 & n+1 \end{vmatrix}$$

$$= (n-1)! \begin{vmatrix} 1 & 0 & \cdots & 0 & -n/2 \\ 0 & 1 & \cdots & 0 & -n/3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -n/(n-1) \\ 1 & 1 & \cdots & 1 & n+1 \end{vmatrix}$$

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$$= n!\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right),$$

$S = n+1+n/2+n/3+\cdots+n/(n-1)$.

It follows that $\frac{D_n}{n!} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, so the sequence $(D_n/n!)$ is unbounded.

B-6. Let $\mathcal{M}$ be a set of real $n\times n$ matrices such that

(i) $I \in \mathcal{M}$, where I is the $n\times n$ identity matrix;

(ii) if $A\in\mathcal{M}$ and $B\in\mathcal{M}$, then either $AB\in\mathcal{M}$ or $-AB\in\mathcal{M}$, but not both;

(iii) if $A\in\mathcal{M}$ and $B\in\mathcal{M}$, then either $AB=BA$ or $AB=-BA$;

(iv) if $A\in\mathcal{M}$ and $A\neq I$, there is at least one $B\in\mathcal{M}$ such that $AB=-BA$.

Prove that $\mathcal{M}$ contains at most n^2 matrices.

Solution It suffices to show that the matrices in $\mathcal{M}$ are linearly independent.

First, observe that for $A, B\in\mathcal{M}$, $AAB = BAA$, by checking the two cases $AB=BA$ and $AB=-BA$. Thus, by (ii) and (iv), $AA=I$ or $AA=-I$. In particular, the matrices in $\mathcal{M}$ are invertible.

To show linear independence, assume that

$$c_1A_1+c_2A_2+c_3A_3+\cdots+c_mA_m=0 \quad (1)$$

for some distinct $A_1, A_2, \ldots, A_m$ in $\mathcal{M}$, with all $c_i\neq 0$ and $m>0$. We may suppose that m is minimal for such relations. Then, multiplying by A_1, we have

$$c_1'I+c_2'B_2+c_3'B_3+\cdots+c_m'B_m=0 \quad (2)$$

where $c_i'=\pm c_i$ and $B_2,\ldots,B_m$ are distinct elements of $\mathcal{M}$, none equal to I. Choosing $C\in\mathcal{M}$ such that $CB_2=-B_2C$ and multiplying equation (2) on the left and right by C^{-1} and C, respectively, we get

$$c_1'I-c_2'B_2+c_3''B_3+\cdots+c_m''B_m=0 \quad (3)$$

with $c_i''=\pm c_i'$ for $i>2$, since $C^{-1}B_iC=\pm B_i$. But then adding equations (2) and (3), we get

$$2c_1'I+(c_3'+c_3'')B_3+\cdots+(c_m'+c_m'')B_m=0,$$

which contradicts the minimality of relation (1).

CORRECTION: Author Larry J. Gerstein points out that in his Note, "*A Reformulation of the Goldbach Conjecture*," this Magazine, Vol. 66, no. 1, February 1993, pp. 44–45, the statement of the twin prime conjecture contains an error. The twin prime conjecture is: There are infinitely many primes p such that $p+2$ is also prime. The Editor regrets the typesetting error.

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